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ISAAC NEWTON, 1642–1727.

IF we seek to determine the birthday of modern science in England, we may find it in 1645, when the group of thinkers who later were chartered as the Royal Society began to meet in the midst of civil war to discuss their problems, or we may be inclined to place it a little earlier, on Christmas Day, 1642, when Isaac Newton was born near Grantham. We cannot now attempt to commemorate that notable date as we should have done in more peaceful times. But Newton, who said " I do not love to be thought to be trifling away my time when I am about the King's business ", would have understood our preoccupation. In his time he saw the advancing aggressions of a dictator expand and prosper, until checked and then shattered by British valour and resolution directed by the genius of John Churchill. We do not need to be reminded of Newton's supreme intellectual powers ; but it may be appropriate now to recall that he was throughout his long life " the unobtrusive but unflinching friend of civil and religious freedom ".

BLITZED LIBRARIES : AN APPEAL.

MANY Libraries have suffered loss through enemy action and will need assistance in reconstituting their collections after the war. It is hoped that owners of mathematical works will co-operate in this task by preserving such volumes as they can spare for this purpose, and by preventing books of value from being pulped for salvage. The London Mathematical Society and the Mathematical Association desire to bring to the notice of readers of the *Gazette* the necessity of providing for the needs of Libraries after the war.

Those who have mathematical works which they are willing to dispose of are asked to communicate with one of the following : the Librarian of the London Mathematical Society (Professor H. Simpson, Bedford College, Springfield, Sidgwick Avenue, Cambridge), the Librarian of the Mathematical Association (Professor E. H. Neville, The Copse, Sonning-on-Thames, Berkshire), or the Editor of the *Gazette* (Professor T. A. A. Broadbent, Wills Hall, Bristol, 9). If a list of books is sent to any of these, the sender will be told which books can safely be pulped. The sender should say if he is willing to allow the remainder to be placed on a list of books which will be available for making up deficiencies in Libraries which have suffered loss. It would be most convenient if the owners would themselves store the books until distribution to Libraries in need can take place ; but it may be possible to arrange for storage in special cases where private storage is difficult.

ON SOME GEOMETRICAL INEQUALITIES.

By D. Pedoe.

In § 1 of this Note we first investigate some geometrical inequalities connecting a set of n unordered points in a plane. An ellipse is associated with the set of points, and use is made of the invariants of this ellipse under rotation of the coordinate axes. Some inequalities, first obtained by Weitzenböck,* are derived and extended. A new geometrical entity, which we call the *mutual area*, is then defined for a set of n and a set of m points with the same centroid. Using the mutual invariants of two ellipses with a common centre under rotations of the coordinate axes, we obtain some inequalities connecting two sets of points with a common centroid.

In § 2 we consider points in three dimensions, and the corresponding extension of our notions, in the course of which we simplify and extend some further theorems of Weitzenböck.† The methods introduced in this Note can be applied to space of any number of dimensions.

§ 1

Suppose we have n points P_i in a plane, not necessarily all distinct, but not all on one straight line. Choosing the origin of rectangular cartesian coordinates at the centroid of these points, let the coordinates of P_i be (x_i, y_i). We define some geometric quantities associated with the n points.

The sum of the squares of the sides of the n-gon formed by the n (unordered) points P_i is S^2, where

$$S^2 = \sum_{i<k} [(x_i - x_k)^2 + (y_i - y_k)^2].$$

Since
$$\Sigma x_i = \Sigma y_i = 0,$$
$$\Sigma x_i^2 + 2\Sigma x_i x_k = \Sigma y_i^2 + 2\Sigma y_i y_k = 0.$$

Therefore
$$S^2 = n(\Sigma x^2 + \Sigma y^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again, the sum of the squares of the areas of the triangles formed by joining pairs of vertices to the origin is f^2, where

$$4f^2 = \Sigma(x_i y_k - x_k y_i)^2 = \Sigma x^2 \Sigma y^2 - (\Sigma xy)^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We now consider the central conic

$$a\xi^2 + 2h\xi\eta + b\eta^2 = 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where
$$a = \Sigma y^2, \quad h = \Sigma xy, \quad b = \Sigma x^2.$$

This conic is invariantly associated with the n-gon under any rotation of the coordinate axes. In fact, the sum of the squared distances of the vertices of the n-gon from any diameter of length $2r$ is precisely $1/r^2$. By virtue of our initial restrictions on the n points P_i it follows that both a and b are always positive, and the conic is therefore an ellipse (the *momental* ellipse).

From (1) and (2) it also follows that both

$$a + b \quad \text{and} \quad ab - h^2$$

are invariants under rotations of the axes. This, of course, is well known. Now, since

$$(a - b)^2 + 4h^2 \geqslant 0,$$

* Weitzenböck, *Math. Zeitschrift*, 5 (1919), (137–146).

† *Loc. cit.*

with equality only when $a=b$ and $h=0$, and this inequality may be written as

$$(a+b)^2-4(ab-h^2)\geqslant 0,$$

we have

$$S^4/n^2-16f^2\geqslant 0.$$

Hence, for any n-gon,

$$S^4\geqslant 16n^2f^2, \quad \ldots\ldots(4)$$

with equality only if the ellipse (3) is a circle. The case $n=3$ is interesting. The inequality becomes

$$(P_1P_2^2+P_2P_3^2+P_3P_1^2)^2\geqslant 48\Delta^2, \quad \ldots\ldots(5)$$

where Δ is the area of the triangle $P_1P_2P_3$. These results are due to Weitzenböck,† but the triangle inequality was rediscovered by the author in an exercise on orthogonal projection.* The inequality (5) becomes an equality if, and only if, the triangle is equilateral. For if we choose the x-axis through the centroid parallel to P_2P_3, say,

$$\Sigma y^2=\tfrac{3}{2}h_1^2,$$

where h_1 is the distance of P_1 from P_2P_3. But Σy^2 is independent of the direction of the x-axis in this case.†

Now consider an m-gon of points P_i', the centroid of these points also being at the origin. For this m-gon we have the geometric quantities :

$$S'^2=m(a'+b'), \quad \ldots\ldots(1')$$

$$4f'^2=a'b'-h'^2, \quad \ldots\ldots(2')$$

and the associated ellipse

$$a'\xi^2+2h'\xi\eta+b'\eta^2=1, \quad \ldots\ldots(3')$$

where

$$a'=\Sigma y'^2, \quad h'=\Sigma x'y', \quad b'=\Sigma x'^2.$$

We define the *mutual area* of our m-gon and n-gon as the sum of the squared areas of the triangles formed by joining every vertex of the one and every vertex of the other to the common centroid ; that is, by M^2, where

$$4M^2=\Sigma(x_iy_k'-x_k'y_i)^2=\Sigma x^2\Sigma y'^2+\Sigma x'^2\Sigma y^2-2\Sigma xy\Sigma x'y'$$
$$=a'b+ab'-2hh'. \quad \ldots\ldots(6)$$

This quantity is, of course, a mutual invariant of the two conics (3) and (3'). We now obtain some inequalities connecting the m-gon with the n-gon.

Consider the expression

$$(a'b+ab'-2hh')^2-4(ab-h^2)(a'b'-h'^2).$$

This is invariant under rotations of the axes, being in fact

$$16M^4-64f^2f'^2.$$

We show that this expression is never negative. Choose the axes to coincide with the principal axes of the ellipse (3). Then our expression becomes

$$(a'b+ab')^2-4ab(a'b'-h'^2),$$

since $h=0$. But this is

$$(ab'-a'b)^2+4abh'^2,$$

and since both a and b are always positive, the expression is either positive or zero. If it is zero,

$$a/a'=b/b' \quad \text{and} \quad h=h'=0$$

* Pedoe, *Math. Gazette*, 25 (1941), 224.

† Cf. Weitzenböck, *loc. cit.* ; and Pedoe, *loc. cit.*

simultaneously. Hence we have proved that

$$M^4 \geqslant 4f^2f'^2,$$

or
$$M^2 \geqslant 2ff', \quad \ldots\ldots(7)$$

with equality only if the ellipses (3) and (3′) are similar and similarly situated.

The case of two triangles, $m=n=3$ gives us

$$M^2 \geqslant \tfrac{3}{2}\Delta\Delta', \quad \ldots\ldots(8)$$

but equality does not necessarily involve the similarity of the two triangles considered.*

The mutual area of an m-gon and an n-gon with the same centroid (referred to subsequently as an m-gon and a *concentric* n-gon) is a function of the relative orientation of the m-gon and n-gon with respect to their common centroid. The sine formula for the area of a triangle shows immediately that

$$M^2 = R\sin^2\theta + 2S\sin\theta\cos\theta + T\cos^2\theta, \quad \ldots\ldots(9')$$

where R, S, T are functions of the elements of the m-gon and n-gon only, and θ measures the relative orientation. It follows that for two given concentric figures in various positions the mutual area has both a maximum and a minimum. We now find these values. Consider the expression

$$[a-b-\lambda(a'-b')]^2 + 4(h-\lambda h')^2. \quad \ldots\ldots(9)$$

It is always positive or zero, for all values of λ, and may be written as

$$p\lambda^2 + 2q\lambda + r,$$

where
$$p = (a'-b')^2 + 4h'^2 = \frac{S'^4}{m^2} - 16f'^2,$$

$$r = (a-b)^2 + 4h^2 = \frac{S^4}{n^2} - 16f^2,$$

and
$$\begin{aligned} -q &= (a-b)(a'-b') + 4hh' \\ &= (a+b)(a'+b') - 2(ab' + a'b - 2hh') \\ &= \frac{S^2S'^2}{mn} - 8M^2. \end{aligned}$$

We have already seen that both p and r are not negative. It follows that

$$q^2 \leqslant pr\,;$$

that is,
$$\left(\frac{S^2S'^2}{mn} - 8M^2\right)^2 \leqslant \left(\frac{S^4}{n^2} - 16f^2\right)\left(\frac{S'^4}{m^2} - 16f'^2\right), \quad \ldots\ldots(10)$$

and therefore
$$8mnM^2 \leqslant S^2S'^2 + [(S^4 - 16n^2f^2)(S'^4 - 16m^2f'^2)]^{\frac{1}{2}}, \quad \ldots\ldots(11)$$

and
$$8mnM^2 \geqslant S^2S'^2 - [(S^4 - 16n^2f^2)(S'^4 - 16m^2f'^2)]^{\frac{1}{2}}. \quad \ldots\ldots(12)$$

* Orthogonal projection shows clearly that a triangle is not uniquely determined by the unique ellipse which touches the sides of the triangle at their midpoints. This ellipse has its centre at the centroid of the triangle and the semi-diameters parallel to the sides of the triangle are respectively proportional to those sides. The ellipse is therefore similar to the ellipse (3). We deduce that the shape of a triangle is not determined by that of the central ellipse for unit particles imagined at the vertices, that is, by the shape of the ellipse (3). Similarity of the central ellipses of two triads does not therefore involve the similarity of the triads.

If either of these two inequalities becomes an equality, the expression (9), must be zero, and then

$$\frac{a-b}{a'-b'}=\lambda=\frac{h}{h'}.$$

The axes of the central ellipses (3) and (3′) are then coincident in direction, the equation to the axes of (3), say, being

$$h(\xi^2-\eta^2)=(a-b)\xi\eta.$$

The relative orientation of the m-gon and n-gon is then determined to a right angle. We note, as a verification of this result, that the expression (9′) for M^2 shows that the values of θ which give the maximum and minimum of M^2 differ by a right angle.

§ 2

We now consider n points P_i in three dimensions, with rectangular cartesian coordinates (x_i, y_i, z_i), the origin being chosen at the centroid of the points. We assume that the points do not all lie in one plane. With these n points we associate the quadric

$$a\xi^2+b\eta^2+c\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta=1, \quad \text{.....(1)}$$

where

$$a=\Sigma x^2, \quad b=\Sigma y^2, \quad c=\Sigma z^2,$$
$$f=\Sigma yz, \quad g=\Sigma zx, \quad h=\Sigma xy,$$

the summations extending to all the points P_i. It is clear that if a diameter of this central quadric is of length $2r$, and has direction cosines (l, m, n), the sum of the squared distances of the points P_i from the plane

$$l\xi+m\eta+n\zeta=0$$

is precisely $1/r^2$. It follows that the quadric is invariantly associated with the points P_i under any rotation of the axes, and therefore that a, b, c are always positive, so that (1) represents an ellipsoid. We define three geometric invariants of our three-dimensional n-gon :

$$S^2=\Sigma[(x_i-x_k)^2+(y_i-y_k)^2+(z_i-z_k)^2]$$
$$=n(\Sigma x^2+\Sigma y^2+\Sigma z^2),$$

that is $$S^2=n(a+b+c). \quad \text{.....(2)}$$

$$4f^2=4\Sigma \text{ (area } OP_iP_k)^2, \text{ or}$$

$$4f^2=ab-h^2+bc-f^2+ca-g^2, \quad \text{.....(3)}$$

since the square of the area of a triangle equals the sum of the squares of the areas of its projections on the coordinate planes. Finally, we write

$$36j^2=36\Sigma \text{ (volume } OP_iP_jP_k)^2$$

$$=\Sigma \begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{vmatrix} \begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}$$

$$=\Sigma \begin{vmatrix} \Sigma x_i^2 & \Sigma x_iy_i & \Sigma x_iz_i \\ \Sigma x_iy_i & \Sigma y_i^2 & \Sigma y_iz_i \\ \Sigma x_iz_i & \Sigma y_iz_i & \Sigma z_i^2 \end{vmatrix},$$

the summation sign inside extending over i, j, k, the outer summation sign over all the values of i, j, k. It follows that

$$36j^2 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \Delta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

These three geometrical invariants of our n-gon are known to be invariants of the quadric (1) under rotation of the coordinate axes.

We now proceed to obtain inequalities connecting the geometric invariants of a three-dimensional n-gon. The expression

$$(a-b)^2+(b-c)^2+(c-a)^2+6(f^2+g^2+h^2)$$

is positive or zero. If it is zero the ellipsoid (1) is a sphere. But we may write this expression as

$$2(a+b+c)^2-6\Sigma(ab-h^2).$$

Therefore
$$\frac{S^4}{n^2} \geqslant 3 \,.\, 4f^2,$$

or
$$S^4 \geqslant 12n^2f^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

This inequality was obtained by Weitzenböck * for the case $n=4$ only, and by a method which uses the condition for the reality of the roots of a certain cubic equation. Our method is essentially the same, but omits inessential considerations.

We now choose our coordinate axes to coincide with the principal axes of the ellipsoid (1). Preserving our notation, we now have

$$S^2 = n(a+b+c),$$
$$4f^2 = bc+ca+ab,$$
$$36j^2 = abc,$$

since
$$f=g=h=0.$$

Also the quantities a, b, c are all positive. The theorem of the arithmetic mean therefore shows that

$$(a+b+c)^3 \geqslant 27abc$$

with equality only if the quantities involved are equal, that is, if the ellipsoid is a sphere. But this may be written

$$S^6 \geqslant 27 \,.\, 36n^3j^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

When $n=4$, $j^2=J^2/4$, where J is the volume of the tetrahedron considered, and (6) becomes

$$J^2 \leqslant \frac{1}{2^6 \,.\, 3^5} \,.\, S^6. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This special case of (6) is obtained by Weitzenböck † only with the help of a general theorem on convex bodies. The equality sign is valid in (7) only when our ellipsoid (1) is a sphere. The tetrahedron must then be an equilateral tetrahedron.‡

* *Loc. cit.* † *Loc. cit.*

‡ Let O be the centroid of the four non-coplanar points $ABCD$. We show that if the products of inertia Σyz, Σzx, Σxy of unit particles at the vertices about any three mutually perpendicular planes through O vanish, then $ABCD$ is equilateral. The product of inertia about the plane BCA and a plane through BC perpendicular to this plane equals the product of inertia of a unit particle at D with respect to these

Another inequality may be obtained by considering the arithmetic mean of the positive quantities bc, ca, ab. We then find

$$\tfrac{1}{3}(bc+ca+ab) \geqslant \sqrt[3]{(a^2b^2c^2)},$$

or

$$(bc+ca+ab)^3 \geqslant 27a^2b^2c^2,$$

which in terms of the geometric invariants of the n-gon is

$$(4f^2)^3 \geqslant 27 \cdot 36^2 \cdot j^4. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

For a tetrahedron this is

$$f^3 \geqslant \frac{27\sqrt{3}}{8} J^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

with equality only if the tetrahedron is equilateral.

We now consider a three-dimensional m-gon of points P_i' with coordinates (x_i', y_i', z_i'), the centroid of these points coinciding with that of our n-gon. We define two simultaneous invariants of the two configurations, both of which may be regarded as *mutual volumes*.

Consider the sum

$$\begin{aligned}
&36\Sigma(\text{volume } OP_i'P_jP_k)^2 \\
&\quad = \Sigma[x_i'(y_jz_k - y_kz_j) + y_i'(x_kz_j - x_jz_k) + z_i'(x_jy_k - x_ky_j)]^2 \\
&\quad = \Sigma x'^2\Sigma(y_jz_k - y_kz_j)^2 + \ldots + \ldots \\
&\qquad\qquad + 2\Sigma y'z'\Sigma(x_kz_j - x_jz_k)(x_jy_k - x_ky_j) + \ldots + \ldots \\
&\quad = \Sigma x'^2[\Sigma y^2\Sigma z^2 - (\Sigma yz)^2] + \ldots + \ldots \\
&\qquad\qquad + 2\Sigma y'z'[\Sigma xy\Sigma zx - \Sigma x^2\Sigma yz] + \ldots + \ldots \\
&\quad = a'A + b'B + c'C + 2f'F + 2g'G + 2h'H,
\end{aligned}$$

where

$$A = bc - f^2, \ \ldots, \ F = hg - af, \ \ldots.$$

We have therefore identified one mutual volume of the m-gon and n-gon with one of the mutual invariants of the ellipsoid (1) and the corresponding ellipsoid for the m-gon. We write in the usual notation,

$$\Theta = 36\Sigma(\text{volume } OP'_iP_jP_k)^2 = 36M^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Similarly

$$\Theta' = 36\Sigma(\text{volume } OP_iP_j'P_k')^2 = 36M'^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Now consider the expression

$$\Theta^2 - 3\Delta'\Theta',$$

where Δ is defined in (4). This expression is an invariant under rotations of the coordinate axes, which we choose to coincide with the principal axes of the ellipsoid associated with the m-gon. Then $f' = g' = h' = 0$. We now find

$$\begin{aligned}
\Theta^2 - 3\Delta'\Theta' &= (ab'c' + bc'a' + ca'b')^2 - 3a'b'c'[\Sigma a'(bc - f^2)] \\
&= \Sigma a^2b'^2c'^2 - \Sigma a'^2bb'cc' + 3\Sigma a'^2b'c'f^2 \\
&= a'^2b'^2c'^2\left[\Sigma\left(\frac{a}{a'}\right)^2 - \Sigma\left(\frac{b}{b'}\right)\left(\frac{c}{c'}\right)\right] + 3\Sigma a'^2b'c'f^2 \\
&= \tfrac{1}{2}(a'^2b'^2c'^2)\Sigma\left(\frac{b}{b'} - \frac{c}{c'}\right)^2 + 3\Sigma a'^2b'c'f^2.
\end{aligned}$$

planes. It also equals the product of inertia of four unit particles at O with respect to these planes. It follows that DO is parallel to the plane through BC perpendicular to ABC. Continuing thus, DO must be perpendicular to ABC. Similarly AO is perpendicular to DBC. Therefore BC is perpendicular to the intersection of DOA and DBC; *i.e.* $DB = DC$. Proceeding thus, the result follows.

Since a', b', c' are all positive, this expression is positive or zero, and only zero when

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'},$$

$$f=g=h=0.*$$

We have therefore proved that

$$\Theta^2 \geqslant 3\Delta'\Theta'. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

It follows that $\Theta'^2 \geqslant 3\Delta\Theta$,

and therefore $\Theta^2\Theta'^2 \geqslant 9\Theta\Theta'\Delta\Delta'$.

But by definition both Θ and Θ' are positive. Therefore

$$\Theta\Theta' \geqslant 9\Delta\Delta', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

or $$(36)^2M^2M'^2 \geqslant 9(36)^2j^2j'^2,$$

$$M^2M'^2 \geqslant 9j^2j'^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

For the case of two tetrads, $m=n=4$, this becomes

$$MM' \geqslant \tfrac{1}{4}JJ'. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

D. P.

THE Mathematical Association has recently been asked by the Committee on Curriculum and Examinations, appointed by the President of the Board of Education under the chairmanship of Sir Cyril Norwood, to answer certain questions about the desirable content of the courses of mathematics in the main school in secondary schools and about the amount of school time necessary to cover the courses.

This matter was referred to the Teaching Committee, and, as meetings of that committee are in abeyance, its members were invited to send their opinions in writing to the chairman of the committee. Several members having responded to this invitation, the chairman combined their contributions and sent a lengthy reply to the Curriculum and Examinations Committee. It was explained at the same time that it was impossible to get a general view of the whole Association at the present time.

It does not seem profitable at present to publish details about the questions and answers. But this note is written partly to let members of the Association know what has been done and also because it is likely that there will be fuller discussions on the same subjects after the war.

The procedure that was adopted was approved by the President of our Association and the draft of the reply was submitted to him before it was sent. The Secretary of the Curriculum and Examinations Committee has sent the thanks of his committee to our Association for the replies and has expressed their appreciation. A. ROBSON.

GLEANINGS FAR AND NEAR.

1413. Converse is a geometrical saying which when said the opposite way to its original way generally means the same thing.—From a script. [Per Mr. P. J. Harris.]

* The associated ellipsoids are then similar and similarly situated.

SOME HOMOTHETIC TRIANGLES RELATED TO THE EULER LINE.

By R. H. Cobb.

In this note the triangle formed by the tangents to the circumcircle at the vertices of a given triangle is called the *circumtriangle* of the given triangle; the triangle formed by the points of contact of the incircle with the sides of a given triangle is called the *intriangle* of the given triangle; and the triangle formed by the points of contact of an excircle with the sides of a given triangle is called an *extriangle* of the given triangle.

1. (i) The circumtriangle PQR and pedal triangle DEF of a given triangle are homothetic.
(ii) The centre of similitude lies on the Euler line of the given triangle.
(iii) The circumcentre of the circumtriangle lies on the same Euler line.

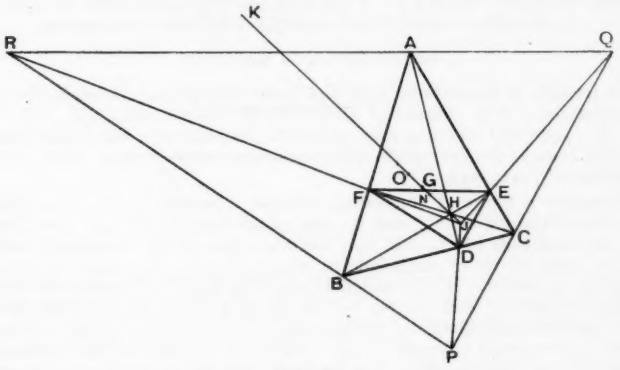


Fig. 1.

For $\angle EDC = \angle A$, from the cyclic quadrilateral $AEDB$, and $\angle PCB = \angle A$ in the alternate segment. Hence PQ is parallel to DE. Similarly QR is parallel to EF, and RP to FD. This proves (i).

Let the centre of similitude be J and the circumcentre of PQR be K. Then since the incentres of the homothetic triangles are collinear with J, O, H and J are collinear, which proves (ii). Also, since the circumcentres and J are collinear, K, N and J are collinear, which proves (iii).

2. (KN, OJ) is harmonic. For, if the ratio of corresponding sides of the triangles PQR, DEF is k, we have

$$k = KJ/NJ = OJ/HJ = (KJ - OJ)/(NJ - HJ)$$
$$= KO/NH = -KO/NO,$$

and so KN is divided harmonically by O and J.

O is the other centre of similitude of the circumcircles of PQR and DEF.

3. The value of k is $1/(2\cos A \cos B \cos C)$. If ABC is acute-angled, it is the intriangle of PQR and k is positive (and lies between 4 and ∞). The triangles PQR and DEF are directly homothetic and the order of points on the Euler line is $KOGNHJ$.

If ABC is obtuse-angled, it is an extriangle of PQR and k is negative (and lies between $-\frac{1}{2}$ and $-\infty$). The triangles PQR and DEF are inversely homothetic. J is between O and H and also between K and N. The order of points on the Euler line depends on the value of k and is shown in the following table.

Value of k	Order of points on the Euler line
$-\infty < k < -2$	$O\ G\ N\ J\ H\ K$
$k = -2$	$O\ G\ N\ J\ H,\ K$ at H
$-2 < k < -1$	$O\ G\ N\ J\ K\ H$
$k = -1$	$O\ G\ N\ H,\ J$ and K at N
$-1 < k < -\frac{8}{9}$	$O\ G\ K\ J\ N\ H$
$k = -\frac{8}{9}$	$O\ G\ J\ N\ H,\ K$ at G
$-\frac{8}{9} < k < -\frac{1}{2}$	$O\ K\ G\ J\ N\ H$

It may be noted that if $k = -1$ the triangles PQR and DEF are congruent. For $k = -1$, the obtuse angle of the triangle ABC must lie between

$$\text{arc cos } \tfrac{1}{2}(1-\sqrt{5}) \text{ and } 135°.$$

4. A triangle is homothetic with the pedal triangles of its intriangle and of its extriangles. The centres of similitude lie on OI, OI_1, OI_2, OI_3, where I, I_1, I_2, I_3 are the incentre and excentres respectively, and these lines are the Euler lines of the intriangle and extriangles respectively. This is simply a corollary of paragraph 1.

5. Consider a figure A consisting of three triangles $\triangle_1$, $\triangle_2$, $\triangle_3$ such that $\triangle_2$ is the intriangle of $\triangle_1$, and $\triangle_3$ the pedal triangle of $\triangle_2$. Let J be the centre of similitude of $\triangle_1$ and $\triangle_2$. Let the figure B be homothetic with the figure A, with centre of similitude S, where $\triangle_1'$, $\triangle_2'$, $\triangle_3'$ and J' correspond to $\triangle_1$, $\triangle_2$, $\triangle_3$ and J. If $\triangle_1'$ coincides with $\triangle_3$, then J, J' and S all coincide. Writing $\triangle_4$ for $\triangle_2'$ and $\triangle_5$ for $\triangle_3'$ we have a figure of five triangles such that $\triangle_2$ is the intriangle of $\triangle_1$, $\triangle_3$ the pedal triangle of $\triangle_2$, $\triangle_4$ the intriangle of $\triangle_3$ and $\triangle_5$ the pedal triangle of $\triangle_4$. Then $\triangle_1$, $\triangle_3$ and $\triangle_5$ are homothetic; $\triangle_2$ and $\triangle_4$ are homothetic; the common centre of similitude lies on the common Euler line of $\triangle_2$ and $\triangle_4$. On this line lie these nine points:

$$O_1,\ O_2 \equiv I_1,\ G_2,\ O_3 \equiv N_3,\ O_4 \equiv I_3 \equiv H_3,\ G_4,\ O_5 \equiv N_4,\ I_5 \equiv H_4,\ J$$

(I_1 is the incentre of $\triangle_1$, and so on).

6. Evidently the series of triangles can be continued indefinitely, $\triangle_{2r}$ being the intriangle of $\triangle_{2r-1}$ and $\triangle_{2r+1}$ the pedal triangle of $\triangle_{2r}$. The odd triangles are all homothetic; the even triangles are all homothetic; the common centre of similitude lies on the common Euler line of the even triangles. On this line lie all the circumcentres and the centroids of the even triangles. A harmonic range is formed by any two consecutive odd circumcentres, the even circumcentre between them, and the centre of similitude.

7. Similar results will hold if each $\triangle_{2r}$ is an extriangle of $\triangle_{2r-1}$ (opposite a specified vertex of $\triangle_{2r-1}$). It is possible in this case for the homothetic triangles to be congruent, if $\triangle_{2r}$ is a triangle for which

$$2 \cos A \cos B \cos C = -1.$$

Then $\triangle_k$ and $\triangle_{k+2}$ are congruent and inversely homothetic. The triangles $\triangle_{k+2}$ and $\triangle_{k+4}$ are congruent and inversely homothetic. Hence $\triangle_{k+4}$ is

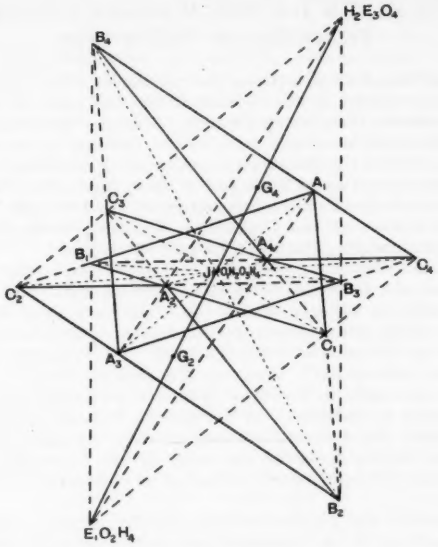


FIG. 2.

identical with $\triangle_k$. The infinite series of triangles thus reduces to four triangles (two congruent pairs), $\triangle_2$ being an extriangle of $\triangle_1$, $\triangle_3$ the pedal triangle of $\triangle_2$, $\triangle_4$ the corresponding extriangle of $\triangle_3$ and $\triangle_1$ the pedal triangle of $\triangle_4$. The points on the Euler line of $\triangle_2$ and $\triangle_4$ are then

$$E_1 \equiv O_2 \equiv H_4,\ G_2,\ O_1 \equiv N_2 \equiv O_3 \equiv N_4 \equiv J,\ G_4,\ H_2 \equiv E_3 \equiv O_4,$$

where E_1 is the appropriate excentre of $\triangle_1$. R. H. C.

1414. *Inertia*, manifesting itself in this tendency of the parts of rotating systems to get farther from the center of rotation, *is called centrifugal force*. . . .

If one whirls a weight around in a circle he finds that he must exert a force inward (centripetal force) to neutralize the centrifugal force urging the weight to fly off at a tangent.—Millikan, Gale and Coyle, *New Elementary Physics*. [Per Mr. R. Smart.]

1415. With cool head and tranquil judgement, imperturbably unconscious of the flight, they oscillated from asymptote to asymptote.—T. E. Lawrence, *Seven Pillars of Wisdom*, Reprint Society Edition, p. 36. [Per Mr. D. J. Finney.]

1416. I am aware that there is such a thing as style in a purely logical argument, and even more perceptibly in the solution of the more abstruse problems of mathematics—Lord Rayleigh's style was elegant, I am told, while Henri Poincaré's had the dazzling brilliance of a flash of lightning—but, having little logic and no mathematics, I am incompetent to discuss these things.—J. Middleton Murry, *The Problem of Style*, p. 73. [Per Mr. A. F. Mackenzie.]

MEMORANDUM ON POST-WAR EDUCATION AND TRAINING OF PHYSICISTS.

THE Institute of Physics has ordered the circulation of a memorandum under the above title, prepared at the request of the Institute by Dr. H. Lowery, Principal of the South-West Essex Technical College. The object is to provoke open discussion on the vital problems of the training of physicists, and the Planning Committee of the Institute hopes to study criticisms and suggestions arising from discussions to be arranged or from direct contributions. These latter may be sent to the Honorary Secretary of the Institute of Physics (Professor J. A. Crowther) at the University, Reading, Berkshire, from whom copies of the memorandum itself can be had on request.

Dr. Lowery has much that is pertinent and important to say on the teaching of physics in schools, technical colleges and universities, particularly in connection with technical colleges, which too often have been regarded as poor relations of the universities. From the *Gazette* standpoint, his most interesting section is that on " Physics and Mathematics " ; he reminds us that the war interrupted our Association's work on the production of a report on the teaching of mathematics to technical students, and expresses the hope that this work may soon be resumed. In the interim, it may be of service to quote brief extracts from this section, intended to show its main drift, and to add some comments, which, however, can only be the expression of a personal opinion and cannot be regarded in any sense as indicative of the views of the Association.

(i) " Mathematics and physics are inextricably joined to one another."

(ii) " The teaching of mathematics has undergone very great changes in the last quarter of a century " and is now " largely dominated by the ' existence theorem ' complex."

(iii) " Physicists must take a hand in deciding the content of the mathematics syllabuses in their courses of training."

(iv) Physicists in training " must cover a wider field than is at present allowed for in the subsidiary mathematics courses in force at the various Universities. It is possible for an honours physics student to take mathematics as his subsidiary subject and yet not to have done any three-dimensional geometry, triple integration, tensor calculus, spherical trigonometry, or vector algebra. . . . I do feel that the present courses in mathematics for physics students in Universities and Colleges omit many of the things that physicists must know if they are to be able to read with understanding the literature of their own subject."

(v) " A new type of examination could be adopted for mathematics for the physicist in which opportunity would be given for the display of a wide knowledge. . . . In many of the Universities the mathematics examination for physics students consists of the papers for a general degree. . . . 10 or 12 questions may be proposed of which it is stated that full marks may be obtained if *about* 7 or 8 are attempted, though it is well known that a student will pass if he does the equivalent of about 1½ to 2 questions. If the examination was designed to test technique then it obviously fails to do so, and it cannot be pretended that it has tested breadth of knowledge, and on either count it is useless from the point of view of the physicist."

We should probably all agree with (i), adding perhaps the proviso that there should be no implication that mathematics in itself is not worthy of attention —from mathematicians. The first part of (ii) is undeniably true, and we may claim that the changes have been for the better ; that the " existence theorem complex " has affected the teaching of mathematics, save to honours students

of that subject, is an assertion less likely to command our assent. But the reproach, if it be one, will best be understood in connection with points (iii) and (v). In (iii) we catch a hint of the suggestion, with which I believe Dr. Lowery would agree, that there is a subject " mathematics ", and there are allied (or derivative) subjects such as " mathematics for physics ", " mathematics for chemistry ", " mathematics for biology ", and so on. In each case the content is a part of the subject " mathematics ", but the form and methods of exposition are—or should be—different. These matters ought to be settled, for " mathematics for physics ", by a collaboration of physicist and mathematician ; the former perhaps to have a deciding vote on content, the latter on form and method of presentation. But we come at once to a difficulty. It seems that Dr. Lowery gives in (iv) a list of mathematical topics which he regards as essential to the training of an honours student of physics, which nevertheless, to his knowledge, are not always included in the subsidiary mathematics course. The mathematician would probably agree at once about the geometry, the triple integration and the vector algebra ; a little less readily about the spherical trigonometry, though this goes to some extent with the geometry ; and with more reluctance to the tensor calculus. The real question is, would the physicists agree? This is neither rhetorical nor reproachful : it is an extremely serious and important question. I have encountered physicists who wanted their students to know some three-dimensional geometry, and others who objected violently to any such teaching ; some who regarded vector algebra as almost essential to a right view of the mathematics of real phenomena, and some who could barely be brought to admit that vector algebra might occasionally be a convenient auxiliary. What is to be done? No doubt the present system, with its local compromises, means that some students learn vector algebra and some do not ; but is it not probable that, under any general scheme narrow enough to command general agreement, none of the subjects mentioned by Dr. Lowery would be included? Perhaps the Institute's circular will give rise to discussions proving this view to be unduly pessimistic. But there is a further point of practical difficulty ; classes are not homogeneous, and in many establishments it would be impossible to arrange for homogeneous classes without an enormous and unthinkable increase in staff. A mathematics class, say in a university, may contain students of " mathematics for physics ", " mathematics for chemistry ", " mathematics for geology ", " mathematics for biology ", as well as students from the Letters side. The biologist may want difference equations, the geologist lattices and groups, the physicist complex variable and operational methods. Even if the " product " of the set, in the language of aggregates, is determinable, how can its presentation best be made vivid? The class is doing differential equations ; but illustrations from dynamics may be utterly obscure to the Arts students, from circuit theory to the geologist. Indeed, if these difficulties were overcome, it seems that the new type of examination outlined by Dr. Lowery in (v) would come of itself. But the outline, which is here quoted almost in full, is too brief to be entirely lucid ; and here again no doubt the Institute is looking for suggestions from readers of the memorandum rather than laying down a policy.

These comments cannot claim to be real criticism, either adverse or constructive. Dr. Lowery has no doubt had such points in mind, and does not need to have them brought to his notice ; many of them are indeed recognised by implication in other sections of the memorandum. They are offered here in the hope that they may persuade teachers of mathematics, particularly those who teach " mathematics for physics ", to obtain, read, discuss and report on this extremely interesting and suggestive memorandum.

T. A. A. B.

MATHEMATICAL NOTES.

1627. *The inflexions of a plane cubic.*

The following proof that a real non-singular cubic has three and only three real inflexions may be of interest.

Since the cubic has nine inflexions, one at least must be real, and so, taking this inflexion to be the point at infinity on the y-axis, we can find a real transformation * to give the equation to the cubic in the form

$$y^2 = u(x) \equiv 4x^3 + qx + r.$$

This leads to

$$\frac{d^2y}{dx^2} = \tfrac{1}{2}u^{-\frac{3}{2}}\,(u_2u_1 - \tfrac{1}{2}u_1^2),$$

where suffixes denote differentiations. The finite inflexions are therefore given by

$$v(x) \equiv u_2u - \tfrac{1}{2}u_1^2 = 0.$$

Now the quartic v is positive as $x \to \pm\infty$ and is negative at its stationary values, for then

$$v_1 = u_3u = 0,$$

i.e. $u = 0$ (since $u_3 = 12$), and so $v = -\tfrac{1}{2}u_1^2$.

Thus, as will be clear from a figure, $v(x)$ has only two real zeros, and at these $v_1(x)$ will have opposite signs. Hence there is only one zero of $v(x)$, say $x = \xi$, which makes u positive and so y real.

There are accordingly only two real finite inflexions, and these lie on a line $x = \xi$ which passes through the real inflexion at infinity, *i.e.* there are just three real inflexions and they are collinear. T. W. CHAUNDY.

1628. *A differential equation with n types of singular solution.*

Those who are interested in the theory of singular solutions of differential equations may like to note an example of a differential equation of order n which, in addition to its general solution, has n distinct singular solutions. The equation is best defined by this general solution which is

$$(n+1)!\; y = (x+A_1)^{n+1} + \ldots + (x+A_n)^{n+1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

so that the differential equation is to be obtained by eliminating the n constants $A_1, \ldots, A_n$ between y and its first n derivatives $y_1, \ldots, y_n$. Evidently

$$(n-r+1)!\, y_r = \sum_{s=1}^{n} (x+A_s)^{n-r+1}.$$

Take $x + A_s$ $(s = 1, \ldots, n)$ to be the roots of an equation

$$z^n + q_1z^{n-1} + \ldots + q_n = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Since $y_n, \ldots, (n+1)!\,y$ are the sums of the first, $\ldots$, $(n+1)$th powers of these roots, we have by Newton's formulae

$$\left.\begin{aligned} (n+1)!\; y + \quad & n!\; q_1y_1 + \ldots + q_ny_n = 0 \\ n!\; y_1 + (n-1)!\; & q_1y_2 + \ldots + nq_n = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ y_n + q_1 & = 0 \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots (3)$$

* See, for instance, H. Hilton, *Plane Algebraic Curves* (Clarendon Press, 1932), 258.

Elimination of $q_1, \dots, q_n$ between these $n+1$ equations gives the differential equation

$$\begin{vmatrix} (n+1)!\ y & n!\ y_1 & \dots & 2!\ y_{n-1} & y_n \\ n!\ y_1 & (n-1)!\ y_2 & \dots & y_n & n \\ \dots & \dots & \dots & \dots & \dots \\ y_n & 1 & \dots & 0 & 0 \end{vmatrix} = 0, \quad \dots\dots\dots\dots\dots(4)$$

the determinant being of order $n+1$.

If now instead of (1) we write

$$(n+1)!\ y = \sum_{s=1}^{m} (x+A_s)^{n+1} \quad (m<n), \quad \dots\dots\dots\dots\dots(5)$$

i.e. if we make some of the roots of (2) zero, Newton's formulae (3) will not be affected, and we shall still arrive at the differential equation (4). Thus (5) also provides a solution of the differential equation. Now, in (5), $y_{n+1}=m$. Thus, if we give m in turn the values $n, n-1, \dots, 1, 0$, we get the general solution (1) of the differential equation followed by n singular solutions, the last being just $y=0$. These are all distinct, *i.e.* one is not included in another for special values of the constants, since they give distinct values of y_{n+1}. I have discussed the matter more fully elsewhere.* T. W. CHAUNDY.

1629. ***The converse of Apollonius' theorem.***

With reference to Mr. A. J. G. May's note 1535 in the *Gazette* for October 1941, the following purely geometrical construction may be of some interest.

Given any triangle ABC and a point D on BC (or on BC produced in either direction) such that $AB^2 - AC^2 = 2BD^2 - 2AD^2$, we produce AD to E so that $AD = DE$.

Then $BE^2 + AB^2 = 2BD^2 + 2AD^2$;

but $AB^2 + AC^2 = 2BD^2 + 2AD^2$ (given).

Hence $BE = AC$.

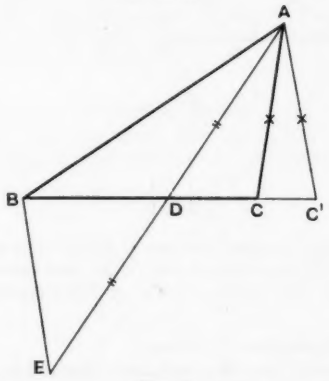


FIG. 1.

Thus *either* D is the midpoint of BC, *or* D is the midpoint of BC', where $AC' = AC$.

* T. W. Chaundy, *Quart. J. of Math.* (Oxford), 12 (1941), 129-47.

Two possible positions of D on BC (or BC produced) exist, except when A lies on the perpendicular through C to BC (in which case C coincides with C').

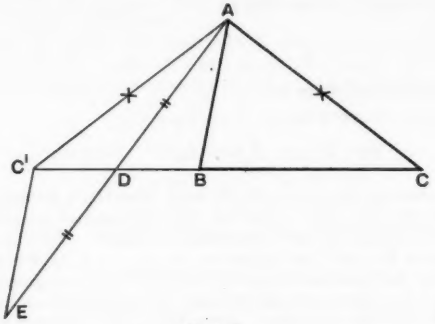


FIG. 2.

Clearly D lies on CB produced (Fig. 2) only if $AC > AB$; and D lies on BC produced (Fig. 3) only if $CX > \frac{1}{2}BC$ (X being the foot of the perpendicular from A to BC).

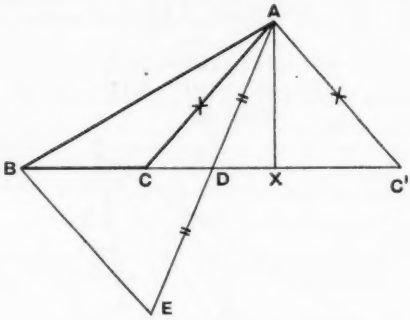


FIG. 3.

Hence two positions of D exist between B and C only when A lies between the perpendiculars to BC through its midpoint, and through the point Y on BC produced such that $CY = \frac{1}{2}BC$, except on the perpendicular through C.

H. F. EVANS.

1630. *On Long Multiplication of Money.*

Mr. R. S. Williamson's note (*Mathematical Gazette*, Dec. 1941, p. 316, Note 1565) on Long Multiplication of Money raises several points of interest.

(i) Is not the best ultimate method for this a practice method?

(ii) As practice is found difficult by young children, is it not desirable that, before they are ripe for practice, they should have a method that is simple and that can become mechanical?

(i) Is not Mr. Williamson feeling after a practice method in the example he gives on p. 317?

Surely the best method of calculating the income tax on £539 at 5s. 6d. in the £ is to take 5s. as $\frac{1}{4}$ of £1 and 6d. as $\frac{1}{10}$ of 5s. That is the method most grown-ups would use and it is essentially a practice method.

Take Mr. Williamson's example, 1s. $3\frac{1}{2}$d. × 539 :

		£	s.	d.
£1 × 539 =	539	0	0	
1s. = $\frac{1}{20}$ of £1	1s. × 539 =	26	19	0
3d. = $\frac{1}{4}$ of 1s.	3d. × 539 =	6	14	9
$\frac{1}{2}$d. = $\frac{1}{6}$ of 3d.	$\frac{1}{2}$d. × 539 =	1	2	$5\frac{1}{2}$
1s. $3\frac{1}{2}$d. × 539 =	£34	16	$2\frac{1}{2}$	

Surely that, as an ultimate method, is simpler than Mr. Williamson's method.

(ii) Mr. Williamson refers to Mr. Webb's method and Mr. Siddons' method. We think it would be better to speak of them as two slightly different arrangements of the same method, the feature of which is that the numbers of £ s. d. are used as multipliers. Incidentally we fail to see that either arrangement lends itself more readily than the other to Mr. Williamson's modification.

This method (with either arrangement) can easily become mechanical, and that is desirable with young children. Dr. Ballard and Mr. Webb have proved that this method is quicker and more accurate than the ten-ten method and it is just as easy to teach.

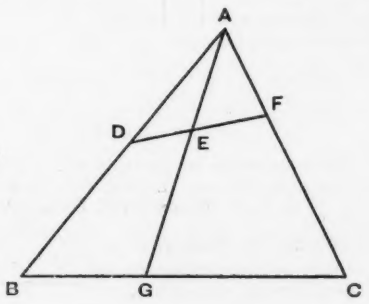
Mr. Webb's arrangement is given in the *Mathematical Gazette* for October 1940, p. 257, and Mr. Siddons' arrangement in the *Mathematical Gazette* for February 1941, p. 36.

We should point out that the example given there (£78 16s. $9\frac{3}{4}$d. × 249) and the examples used in Mr. Webb's tests are much heavier than it is desirable to give to children in the early stages. They were chosen not as examples for teaching but to show the full application of the method and to test the method against the ten-ten method.

A. W. SIDDONS.
H. WEBB.

1631. *Statical solution of a geometrical problem.*

If $AD : DB$, $AF : FC$ and $DE : EF$ are given rational values (say, 3 : 2, 1 : 2, 1 : 3), to calculate the values of $BG : GC$ and $AE : EG$.



$3+2=5$, $1+2=3$, $1+3=4$, and the least common multiple of 5, 3 and 4 is 60.

A mass $60m$ at E can be replaced by $45m$ at D and $15m$ at F. These can be replaced by $28m$ at A, $27m$ at B and $5m$ at C, and these by $28m$ at A and $32m$ at G.

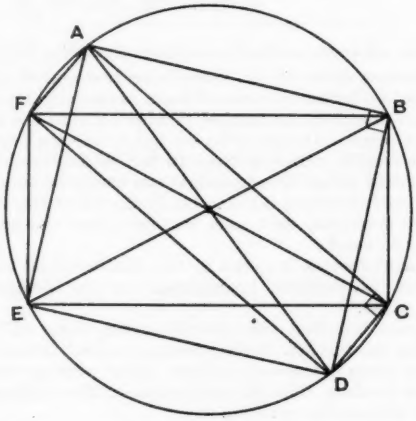
Hence $$BG : GC = 5 : 27,$$

$$AE : EG = 32 : 28 = 8 : 7.$$ S. INMAN.

1632. *The rational cyclic hexagon.* (See Notes 1514, 1557).

The problem is to find a general formula for a cyclic hexagon with diameters for diagonals, all of whose sides are rational.

Let the diameters AD, BE, CF be of unit length : and let AB, CD have rational values such that AC, BD are also rational.



By Ptolemy's theorem, $BC = AC \,.\, BD - AB \,.\, CD$, and AB, AC, BD, CD, are rational. Hence BC is rational.

Also $$BD \,.\, EC = CD + BC \,.\, DE = CD + AB \,.\, BC,$$

and so EC is rational.

Thus all the sides and diagonals are rational.

To satisfy the original assumption we may take

$$AB = (\lambda^2 - 1)/(\lambda^2 + 1), \quad CD = (\mu^2 - 1)/(\mu^2 + 1);$$

whence $$BC = \frac{(\lambda+\mu)^2 - (\lambda\mu - 1)^2}{(\lambda+\mu)^2 + (\lambda\mu - 1)^2}.$$

Multiplication by the least common multiple of the denominators will give integral values for the results, for any rational values of λ and μ.

H. P. F. SWINNERTON-DYER (Per W. HOPE-JONES).

1633. *On "Al Capone and the Death Ray".*

In reply to the request by the author (*Gazette*, XXV, December 1941) for "an easier way", the following is suggested.

Throughout what follows, the coordinates of all points are integral.

Let the number of unrestricted paths from any point S to any point R by unit step progressions either due East or due North be denoted by (SR), where, of course,

$$(SR)=\binom{p+q}{p},$$

p and q being positive relative coordinates of R with reference to S.

Let the point $(1, -1)$ be I, the point $(0, 0)$ be O, the point $(n, mn+c)$ be Q, and the point $(r, mr-1)$ be P_r. Let the number of paths from P_r to Q which meet the line $y=mx-1$ but only at P_r be denoted by $[P_rQ]$.

Now for all values of r,

$$\binom{mr+r-1}{r}\equiv m\binom{mr+r-1}{r-1},$$

and so

$$(OP_r)=m(IP_r),$$

and hence

$$\sum_1^n(OP_r)\,[P_rQ]=m\sum_1^n(IP_r)[P_rQ]$$
$$=m(IQ).$$

But the left-hand side is the number of paths from O to Q which meet the line $y=mx-1$ or the number that cross the line $y=mx$. Hence the number that do not cross $y=mx$ is

$$(OQ)-m(IQ)=\frac{c+1}{mn+c+1}\binom{mn+n+c}{n}.$$

The special case $m=1$ admits the use of a real image instead of the pseudo-image I. The proof, which appeared in the *Gazette* (XVIII, May 1934, Note 1103, p. 124) is both simpler and gives a more general result. Other problems which can be treated similarly are illustrated by the following, a solution of which appeared in the *Chess Amateur*: "On a chessboard, the total number of routes by which the two wing pawns may proceed to queen is equal to the total number of routes by which the two royal pawns may proceed to queen on the two royal files", with a generalisation to a board of $m\times n$ squares.

Is it too fanciful to trace an analogy here to Kelvin's use of images in electrostatics?

J. PEACOCK.

1634. *A special quadrilateral.*

In a recent note on rational quadrilaterals, a doubt was expressed as to the possibility of constructing a quadrilateral cyclic, circumscribable, of rational area, sides and diagonals. An infinity can be obtained as follows:

Let $ABCD$ be the quadrilateral, with $AB=u$, $BC=v$, $CD=x$, $DA=y$, and $\cos ABC=k$. Then if $ABCD$ is cyclic,

$$u^2+v^2-2uvk=AC^2=x^2+y^2+2kxy. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If $ABCD$ is also circumscribable, then

$$x=v+y-u,$$

and substituting in (1) we have

$$(v+y-u)^2+y^2+2k(v+y-u)\,y=u^2+v^2-2kuv.$$

This reduces to

$$y^2+y(v-u)+(k-1)uv/(k+1)=0.$$

If y is to be rational,

$$v^2+u^2-2(3k-1)uv/(k+1)$$
$$=\text{square}=(v+pu)^2, \text{ say};$$

and for AC to be rational,

$$v^2+u^2-2uvk=\text{square}$$
$$=(v-pu)^2, \text{ say.}$$

Subtracting, $$p=(k-1)^2/2(k+1).$$

Again, $$v^2+u^2-2uvk=v^2-2puv+p^2u^2$$

or $$u-2vk=p^2u-2pv.$$

This is satisfied if $u=2(k-p)$ and $v=1-p^2$.

Incidentally a similar method can be used for

$$s^2+as+b=\text{square},$$
$$s^2+cs+d=\text{square},$$

by putting $s=t+n$, and choosing n so that the absolute terms are equal.

Continuing, $$y=\tfrac{1}{2}\{-(v-u)+v+pu\},$$

taking the positive root of the discriminant, and so

$$y=\tfrac{1}{2}u(1+p),$$
$$x=v+y-u,$$
$$AC=v-pu.$$

The area is $\frac{1}{2}(uv+xy)\,.\,\sqrt{(1-k^2)}$ and is rational if $k=2q/(1+q^2)$, where q is rational. The diagonal BD is, by Ptolemy's theorem, $(ux+vy)/AC$ and so is rational.

If $q=3$, then $k=3/5$, $p=1/20$, $u=11/10$, $v=399/400$, $y=231/400$, $AC=377/400$.

This gives the quadrilateral $u=440$, $v=399$, $y=231$, $x=190$, $AC=377$, $BD=6061/13$; or finally we have the integral quadrilateral

$$AB=5720,\ BC=5187,\ CD=2470,\ DA=3003,\ AC=4901,\ BD=6061.$$

J. PEACOCK.

1635. *The limit of* $\{\Delta^n f(x)\}/(\Delta x)^n$.

If $$\Delta f(x)=f(x+\Delta x)-f(x)$$

and $$\Delta^{n+1}f(x)=\Delta\{\Delta^n f(x)\},$$

so that $\Delta^n f(x)$ is the nth finite difference of the function $f(x)$, then it is well known that the limit of $\Delta^n f(x)/(\Delta x)^n$, as Δx tends to zero, is $f^{(n)}(x)$, the nth derivative of $f(x)$; yet I cannot recall having seen an *elementary* proof of this important result in any of the standard textbooks. The proof given by de la Vallée Poussin in his *Cours d'Analyse* (7th edition, 1930, I, p. 73) is subtle but appeals to the *mean value theorem*, as also does the proof in Goursat's *Cours d'Analyse* (2nd edition, vol. I, p. 50); the latter, moreover, is fallacious since it ignores the fact that the value of θ for which $f(x+h)-f(x)=hf'(x+\theta h)$ depends upon both x and h. In Hardy's *Pure Mathematics* the theorem is given as an example, with the unnecessary condition that $f^{(n)}(x)$ is continuous, and in Boole's *Calculus of Finite Differences* the truth of the result is observed in a few particular instances, but no general demonstration is offered.

The following is a straightforward proof from first principles. We observe first that $\Delta^n f(x)$ satisfies :

$$\Delta^{n+1} f(x) = \Delta^n \{\Delta f(x)\}$$
$$= \Delta^n f(x + \Delta x) - \Delta^n f(x) \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$f(x + n \,.\, \Delta x) = (1 + \Delta)^n f(x) \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and since $\Delta^n = (1 + \Delta - 1)^n$

$$= (1 + \Delta)^n - \binom{n}{1}(1 + \Delta)^{n-1} + \binom{n}{2}(1 + \Delta)^{n-2} + \ldots + (-1)^n,$$

it follows from (2) that

$$\Delta^n f(x) = f(x + n \,.\, \Delta x) - \binom{n}{1} f(x + \overline{n-1}\,\Delta x) + \binom{n}{2} f(x + \overline{n-2}\,\Delta x)$$
$$- \ldots + (-)^n f(x). \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Proof of (2). For $n = 1$ the result stated is equivalent to the definition of $\Delta f(x)$; if it is true for $n = k$, then

$$f(x + \overline{k+1}\,\Delta x) = f(x + \Delta x + k \,.\, \Delta x)$$
$$= (1 + \Delta)^k f(x + \Delta x)$$
$$= (1 + \Delta)^k (1 + \Delta) f(x)$$
$$= (1 + \Delta)^{k+1} f(x),$$

which proves (2) for $n = k + 1$ and hence (2) is true for all n.

Denote the rth derivative of $f(x)$ by $f^{(r)}x$ and the rth derivative of $\phi(x, \Delta x)$ with respect to Δx by $D_\Delta^r \phi$. Then by (3), since

$$\binom{n}{r}(n - r) = n\binom{n}{r} - n\binom{n-1}{r},$$

we have $D_\Delta \Delta^n f(x) = n f'(x + n \,.\, \Delta x) - \binom{n}{1}(n-1) f'(x + \overline{n-1}\,\Delta x)$

$$+ \binom{n}{2}(n-2) f'(x + \overline{n-2}\,\Delta x) - \ldots$$
$$= (n\,\Delta^n + n\Delta^{n-1}) f'(x).$$

Suppose that, for $r \leqslant n$,

$$D_\Delta^r \Delta^n f(x) = [n^r \Delta^n + a_1 \Delta^{n-1} + a_2 \Delta^{n-2} + \ldots + \{n!/(n-r)!\}\Delta^{n-r}] f^{(r)}(x), \quad \ldots\ldots (4)$$

where each a_r depends only on r and n.

This is true for $r = 1$. If it is true for $r = 1, 2, 3, \ldots, k$ with $k < n$, then

$$D_\Delta^{k+1} \Delta^n f(x) = D_\Delta \{D_\Delta^k \Delta^n f(x)\}$$
$$= D_\Delta (n^k \Delta^n + a_1 \Delta^{n-1} + \ldots) f^{(k)}(x)$$
$$= [n^{k+1} \Delta^n + \{n^{k+1} + (n-1)a_1\} \Delta^{n-1} + \{(n-1)a_1 + (n-2)a_2\} \Delta^{n-}$$
$$+ \{(n-2)a_2 + (n-3)a_3\} \Delta^{n-3} + \ldots$$
$$+ \{n!/(n-k-1)!\} \Delta^{n-k-1}] f^{(k+1)}(x),$$

which proves the result for $r = k + 1$ and $k + 1 \leqslant n$.

Thus (4) is true for any n and $r \leqslant n$.

It follows from (3) that, for any $g(x)$,

$$\{\Delta^n g(x)\}_{\Delta x=0} = \left\{1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-)^n\right\} g(x)$$
$$= (1-1)^n g(x)$$
$$= 0, \quad \text{provided } n > 0;$$

if $n = 0$, $\{\Delta^n g(x)\}_{\Delta x=0} = \{g(x)\}_{\Delta x=0}$
$$= g(x).$$

Hence from (4)

$$\left.\begin{aligned} \{D_\Delta^r \Delta^n f(x)\}_{\Delta x=0} &= 0, && \text{if } r < n \\ &= (n!) f^{(n)}(x), && \text{if } r = n \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Next we show that, if for some function $\phi(x)$,

$$\phi(0) = \phi'(0) = \ldots = \phi^{(n-1)}(0) = 0, \ \phi^{(n)}(0) = 2\alpha > 0,$$

then
$$\phi(h)/h^n > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

for all h such that $|h| \leqslant h_\alpha$ where $h_\alpha \to 0$ as $\alpha \to 0$.

Proof of (6). Since

$$\{\phi^{(n-1)}(h) - \phi^{(n-1)}(0)\}/h \to \phi^{(n)}(0),$$

we have
$$|\phi^{(n)}(0) - \{\phi^{(n-1)}(h)/h\}| < \alpha,$$

provided that $|h| \leqslant h_\alpha$ where $h_\alpha \to 0$ as $\alpha \to 0$;

that is,
$$\phi^{(n-1)}(h)/h > \alpha.$$

Thus for $0 \leqslant h \leqslant h_\alpha$,
$$\phi^{(n-1)}(h) \geqslant h\alpha$$

and
$$-\phi^{(n-1)}(-h) \geqslant h\alpha.$$

Hence by repeated integration we find, for $0 \leqslant h \leqslant h_\alpha$,

$$\phi(h) \geqslant \alpha h^n/n!,$$
$$(-)^n \phi(-h) \geqslant \alpha h^n/n!,$$

and therefore
$$\phi(h)/h^n \geqslant \alpha/n!, \text{ for } |h| \leqslant h_\alpha.$$

Similarly we can show that if

$$\phi^{(r)}(0) = 0, \ r < n, \text{ and } \phi^{(n)}(0) = -2\alpha < 0,$$

then
$$\phi(h)/h^n < 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6')$$

for all h such that $|h| \leqslant h_\alpha$.

Let
$$\phi(\Delta x) = (\Delta x)^n \cdot \{f^{(n)}(x) + \lambda\} - \Delta^n f(x).$$

Then $\phi(0) = 0$ and by (5)

$$\{D_\Delta^r \phi(\Delta x)\}_{\Delta x=0} = 0, \quad r < n$$
$$= (n!)\lambda, \quad r = n.$$

Hence by (6) if $\lambda = \epsilon > 0$, $\phi(\Delta x)/(\Delta x)^n > 0$ and so

$$\{\Delta^n f(x)/(\Delta x)^n - f^{(n)}(x)\} < \epsilon, \text{ for } |\Delta x| \leqslant h_\epsilon,$$

and by (6′) if $\lambda = -\epsilon < 0$,

$$\{\Delta^n f(x)/(\Delta x)^n - f^{(n)}(x)\} > -\epsilon, \text{ for } |\Delta x| \leqslant h_\epsilon.$$

Whence, since $h_\epsilon \to 0$ as $\epsilon \to 0$, we have

$$\lim_{\Delta x \to 0} \Delta^n f(x)/(\Delta x)^n = f^{(n)}(x).$$

R. L. GOODSTEIN.

REVIEWS.

The Philosophy of Alfred North Whitehead. Library of Living Philosophers, Vol. III. Edited by PAUL ARTHUR SCHILPP. Pp. xviii, 745. $4. 1941. (Northwestern University, Evanston and Chicago)

This book is the third in a very excellent series devised on an original plan. The general idea is to collect a number of essays from different writers on the various aspects of the work of an eminent living philosopher, to submit them all to him, and then to publish them in a single volume together with an essay by him in reference to their comments and criticisms. Advanced age and a severe illness have unfortunately prevented Whitehead from contributing a terminal essay, but he has supplied a very attractive short chapter of autobiography at the beginning, and has allowed his two most recent philosophical papers—*Mathematics and the Good* (1939) and *Immortality* (1941)—to be printed at the end. The volume is provided with an admirable bibliography of Whitehead's published writings in chronological order, with a selection of the more important reviews of them. It covers 21 pages and contains 92 entries.

Essays are contributed by eighteen authors. The first and much the longest (100 pages) is an elaborate account of the development of Whitehead's philosophy by Mr. Victor Lowe. Then come five papers on the more purely scientific aspects of Whitehead's work (*Quine* on his contributions to logic, *Northrop* on his philosophy of science, *McGilvary* on his doctrine of space-time, *Needham* on the biological bearings of the system, and *Hughes* on his psychological doctrines). These are followed by five essays on Whitehead as a speculative philosopher and metaphysician (*Urban* on his philosophy of language in relation to his metaphysics, *Ritchie* on his defence of speculative reason, *Murphy* on his method of doing speculative philosophy, *Hocking* on his doctrine of mind and nature, and *Sellars* on his philosophy of organism and its relation to physical realism). Three contributions deal with various aspects of Whitehead's theories of ethics and aesthetics (*Goheen* on his theory of value, *Morris* on his aesthetics, and *Schilpp* on his moral philosophy). Two papers are concerned with Whitehead's theology (*Bixler* on his philosophy of religion, and *Hartshorne* on his idea of God). There is one essay, by Holmes, on Whitehead's views on education; and a final essay, by Dewey, on the philosophy of Whitehead.

It is plainly useless to attempt in a few lines to give a synopsis of a book of 750 pages of very varied and highly complicated matter. I will therefore confine myself to mentioning those parts which are likely to be of most interest to readers of the *Mathematical Gazette* and concluding with a few very general remarks.

Mr. Lowe should earn the thanks of mathematicians by giving a very full account of Whitehead's first book, *Universal Algebra*, which has been but little read. Mr. Quine, moreover, gives a critical account of that considerable part of it which deals with the algebra of Boolean logic. I believe that Whitehead, in his *Universal Algebra*, is the only English mathematician (and one of the very few mathematicians in the world) to take Grassmann's *Ausdehnungslehre* seriously and to apply its methods on a large scale. He also made fundamental contributions to the technical side of the algebra of logic. Another extremely important and most fascinating early work of Whitehead's to which Mr. Lowe calls attention is his long paper on "Mathematical Concepts of a Material World" in the *Philosophical Transactions of the Royal Society* for 1906. My own attention was first called to it by Bertrand Russell when I was an undergraduate. I was enormously impressed by it at the time, and

on looking at it again lately the impression of its great originality and technical skill was repeated. Yet, so far as I know, it had no influence whatever. I have the same uneasy feeling about Whitehead's own version of the general theory of relativity, which he worked out in his book, *The Principle of Relativity* (1922), in conscious opposition to what he regarded as fundamental flaws in Einstein's conceptions. I suspect that hardly any mathematician who is competent to criticise it has ever taken the trouble to read it. (I must except Professor Eddington, for he tells me that he has explicitly dealt with certain parts of it in the *German*, though not in the English, edition of his *Mathematical Theory of Relativity*.)

Mr. Quine gives a very clear critical account of Whitehead's contributions to logic from the Boolean stage of *Universal Algebra*, through the period when he co-operated with Russell in writing *Principia Mathematica*, to his paper on " Classes, Numbers, and Validation " in *Mind* of July 1934. In the course of this he expresses the opinion that certain obscurities in *Principia Mathematica* would have been avoided if the authors had paid more attention to some of the distinctions which Frege drew in his logical work.

Of the power, acuteness, and originality of Whitehead's contributions to mathematical logic there can be no doubt. It also seems to me obvious that the group of books *Concept of Nature*, *Principles of Natural Knowledge*, and *Principle of Relativity* are enormously important detailed attempts to show the connections between the refined concepts of mathematical physics and the crude data of sense-perception. I find it much harder to make up my mind about the later and more purely metaphysical works, starting with *Process and Reality*. In the first place, I must confess that I do not fully understand one sentence in ten of them, and that I can seldom see any conclusive reason to accept those fragments which I think I can understand. Again, it seems to me that at times Whitehead is simply enunciating vague platitudes, which might amount to anything or nothing, in a complicated technical terminology.

But, having said so much, I would conclude with the following remarks on the other side. No one who has known Whitehead could possibly believe that he is ever deliberately playing the oracle to audiences of half-baked intellectuals. Anyone, who recognises as I do, the immense power and originality of those parts of Whitehead's work which are intelligible will be well advised to suspect that what he cannot understand and what appears to him to be platitude may contain important and unrecognised truths. Again, it seems to me that Whitehead is always thinking and talking about the problems which really are fundamental in philosophy ; and I find that, after reading him, I am often stimulated to think of them in new ways, which have certainly been suggested by what I have read, though I am not at all sure whether I have understood what he has been trying to convey. Lastly, Whitehead has the immense advantage over many contemporary philosophers who approach philosophy from the side of mathematics and physics that he is a man of immensely wide culture and interests, who thus possesses what they so lamentably lack, viz. an adequate factual and emotional basis on which to erect a system of philosophy which is not childishly naïve or grotesquely inadequate to the facts of life. Even the most obscure of his writings are constantly illuminated by remarks of penetrative insight and wisdom, and these are often couched in extremely happy and striking phrases.

Though all the writers in the book under review are plainly impressed with Whitehead's achievements, several of the essays are highly critical and raise very pertinent questions both of principle and of detail. In this respect I found the papers by Northrop, McGilvary, Urban, Murphy, and Sellars (in addition to that by Quine already mentioned) most interesting.

The book is excellently printed and produced. I have noticed only four places where it seems to me that there are misprints. On p. 54, line 4, should not " bases " be substituted for " biases "? On p. 167, line 8, correct " *Geometrical* " to " *Universal* ". On p. 334, line 9, correct " rhapsody " to " rhapsodist ". Lastly, on p. 369, line 12, substitute " of " for " for ".

C. D. BROAD.

The Development of the Sciences (Second Series). By OYSTEIN ORE and OTHERS. Pp. vii, 336. 18s. 6d. 1941. (Yale University Press; Humphrey Milford)

Writers on the history of science must always face an awkward dilemma of arrangement. Is a chapter to deal with one subject over a long period, or simultaneously with all subjects? A writer who sets out to cover the whole field will probably be interested in the relationships between the sciences, their comparative development, and their relation to the history of the times. If so, he will wish to adopt the second method, in so far as it is practicable. But the difficulties are considerable and he must inevitably compromise. The other method is less ambitious, easier, and can be adopted in extreme form, as when different authors write on the different subjects. The reader is then left to make his own comparisons.

In this series of eight lectures by professors of Yale University, one is given a rapid review of each subject in turn, and a broad comparison of one with another is both natural and easy. One cannot fail to notice, for example, the parallel development of mathematics and astronomy, and, on the other hand, the comparatively late development of chemistry and the biological sciences.

Professor Oystein Ore deals with the history of mathematics in 51 pages. The earlier part of this chapter is a very successful summary, in which he is able to discuss the broad outlines of ancient mathematics and at the same time to find space for some selected detail, as, for instance, when he tells us of the recent advances in our knowledge of Babylonian mathematics. It is fairly easy to treat ancient mathematics in this way, but when it comes to the Renaissance it is almost impossible to avoid mentioning a multitude of names, and perhaps it would be wrong not to do so. In the modern period this difficulty becomes greater, and the mere variety of subjects makes it hardly possible to avoid a kaleidoscopic effect. From Gauss onwards he reviews, in 24 pages, the whole field of modern research, including projective geometry, theory of equations, groups, the fate of the three classical construction problems, non-euclidean geometry, theory of numbers, function theory, probability, and the logical foundations. The only wonder is that it should be so readable. To give honour where honour is due would be impossible in so small a space, but it is worth noting that he describes Hamilton and Grassmann (p. 42) as having made " preliminary studies " of vectors, a reflection, perhaps, of the American preference for the notation of Gibbs over that of Grassmann.

The value of this short summary of the history of mathematics is that it gives an impression of the whole, such as the ordinary reader can hardly obtain from a larger work. The later pages, too, give a vivid impression of the aliveness of the subject.

The writer on astronomy had perhaps an easier task. He shows how astronomy began with time-reckoning, and then follows a history closely parallel to that of mathematics as far as the Renaissance. He continues with spectroscopy, photography, the study of the sun, and refers somewhat briefly to modern work on the stars.

The chapter on physics, which occupies 30 pages, begins less happily with

a dramatic account of Giordano Bruno's martyrdom. Galileo, though he died peaceably, would perhaps have been a better choice to mark the beginning of modern physics. The same tendency to dramatisation appears later in the remark (p. 105) that the rise of materialism is definitely traceable to Helmholtz's lecture in 1847 to the Physical Society in Berlin. The chapter consists of a chronological summary up to the year 1600, followed by sections on "Mechanics and Heat", "Electricity and Magnetism", "Light and Relativity", "Spectroscopy and Atomic Structure", "Elementary particles and Nuclear Structure". Each of these inevitably contains a rather large number of facts, and in general it may be said that in a book of this sort a broad treatment is more helpful to the reader than that of the writer who, like some amateur photographers, wants to "get it all in".

The rest of the book is divided, rather unevenly, between chemistry (25 pages), geology (50 pages), biology (50 pages), psychology (44 pages), and "medicine and the sciences" (26 pages). In the last of these, Professor J. F. Fulton wisely confines himself to three themes, the influence of medical studies on the chemistry of gases, on the discovery of electrical potential, and on thermodynamics. He is thus able to adopt a more leisurely style which is refreshing to the reader after travelling so far and so fast. E. H. L.

THE DECIMAL SUBDIVISION OF THE DEGREE.

Manual of Gear Design. Section One. Eight Place Tables of Angular Functions in Degrees and Hundredths of a Degree and Tables of Involute Functions, Radians, Gear Ratios and Factors of Numbers. By EARLE BUCKINGHAM. 184 pages. $8\frac{1}{2}'' \times 11''$. 15s. 1935. (Machinery, New York)

Of the three main systems of subdivision of the quadrant the commonest and best established is the sexagesimal. As L. J. Comrie says: * "In this country the sexagesimal system does not even seem to be dying a hard death, but is still flourishing healthily, so . . . cannot be ignored." This quotation suggests, correctly, that Comrie is aware of drawbacks in the sexagesimal system, as are many, perhaps most, mathematicians.

The most recently introduced subdivision, the centesimal, *i.e.* the decimal division of the quadrant, is also quite well represented amongst modern published tables of trigonometrical functions. Introduced during the French revolution, it was favoured by E. Sang, who said in 1890: † "The change to this system is inevitable. Each new discovery, each improvement in the art of observing intensifies the need for the change, at the same time that each augmentation of our stock of data arranged in the ancient way adds to the difficulties. How much the change is needed may be estimated by an inspection of the *Nautical Almanac*. Every page in it cries out aloud in distress 'Give us decimals'. . . . The labour of interpolation is unnecessarily doubled at the very least, and that heavy burden is laid on the shoulders of all the daily users of the ephemeris . . . and this continuous waste of labour has much need to be ended." It is worth noting that, since its remodelling by Comrie in 1931, the *Nautical Almanac* has, in fact, given the fractional part of the degree in many of its columns of angles in both sexagesimal and decimal form.

The arguments for the third system of subdivision, into degrees and decimals of a degree, seem of considerable strength. J. W. L. Glaisher said: ‡

* *The Twin Marchant Calculating Machine and its Application to Survey Problems*, p. 20. London, Scientific Computing Service. 1942.

† Quoted by C. G. Knott, *Napier Tercentenary Handbook* (edited by E. M. Horsburgh), p. 42. Roy. Soc. Edin. 1914.

‡ *Report on Mathematical Tables.* B.A. Report, 1873, p. 64, footnote.

"The centesimal division of the degree is of paramount importance, whereas the centesimal division of the right angle is of next to none at all; . . . The hundredth part of a right angle is almost as arbitrary a unit as the ninetieth; and no advantage (but on the contrary great inconvenience) would result from the change: but . . . to measure angles by degrees and decimals of a degree, would ensure all the advantages of a decimal system (a saving of work in interpolations, multiplications, etc.). This Briggs and his followers, Roe, Oughtred, John Newton, etc., perceived and acted upon two hundred and fifty years ago; . . . It may be taken for granted that the magnitude of the degree will never be altered; but there is no reason why sexagesimal minutes and seconds should not be replaced by decimals of a degree; and this is a change which might, and we hope will hereafter be made."

Again, J. R. Milne said:* "Historically, the most important system, of course, is the sexagesimal, and to this system many computers, such as astronomers and surveyors, are tied, because of their records and the graduations of their instruments. Such a restriction, however, does not obtain in the case of an increasing number of people who have to perform calculations involving angles, and for them the question is purely one of utility. Unfortunately, owing to non-agreement as to the fundamental unit, the application of decimalisation has led to the introduction of three different systems of units, and the publication of corresponding tables. In one case the whole circumference of the circle has been taken as the unit to be decimally subdivided; in another, the quadrant; and in a third, the existing degree, which is a ninetieth of the quadrant. As regards the first two of these systems, although each of them—and more especially the second—has certain advantages, it may be doubted whether these are sufficiently great to justify that entire departure from the old units which the systems entail. On the other hand the decimal subdivision of the sexagesimal degree presents all the advantages of a decimal system, while conserving the existing unit. Many existing tables in the sexagesimal system already take six minutes as the advance of their argument, thus dividing the degree decimally. That some decimal system will ultimately triumph cannot be doubted, but this consummation, so much to be desired, may be unduly delayed by the conflicting claims of rival systems; and surely it would be wise in a case of this kind, when the obstacles to be overcome are so formidable, to take the line of least resistance by retaining the existing unit."

It should perhaps be noted that, when we consider trigonometrical functions as particular cases of Jacobian elliptic functions, with $k=0$, the centesimal subdivision is important; for any elliptic function it is often desirable to subdivide the quarter-period K, and it is largely due to the case $K=\frac{1}{2}\pi$, *i.e.* $k=0$, that the question of a subdivision other than centesimal arises. The functions $\sin \frac{1}{2}\pi x$, $\cos \frac{1}{2}\pi x$ have, in fact, an importance of their own. This consideration weakens to some extent the arguments of Glaisher and Milne against the centesimal subdivision of the quadrant without, as it seems to the writer, unduly weakening those for the decimal subdivision of the degree.

One further argument in favour of the existing degree should be noted; this lies in the importance of the angles 15°, 30°, etc., which are very inconveniently represented in centesimal "degrees", or grades. In addition the convenience of 360° for subdivision in connection with harmonic analysis should be noted.

In spite of the arguments outlined above, there remain serious gaps in the tabulation of natural trigonometrical functions with degree-decimal arguments.

* *Napier Tercentenary Memorial Volume* (edited by C. G. Knott), p. 313. London, Longmans. 1915.

Trigonometria Britannica, by Henry Briggs, with explanatory and other matter by H. Gellibrand, appeared in 1633. This gives sines to 15 decimals, and tangents and secants to 10 decimals for every hundredth of a degree. W. Oughtred in 1657 published 7-place tables of the same functions for the same arguments. Briggs also gives log sin to 14 decimals and log tan to 10; tables of logarithmic values were also given by Oughtred, Roe and Wingate, and John Newton. These tables are, of course, virtually unobtainable nowadays.

Modern tabulations of natural values include *Siebenstellige Werte der trigonometrischen Funktionen* . . . by J. Peters, 1918. This gives sine and tangent for every thousandth of a degree from 0° to 90°, in separate tables—an inconvenient arrangement. In 1937 Peters published his *Sechsstellige Werte der Kreis- und Evolventen-Funktionen* . . . giving six functions for every hundredth of a degree from 0° to 45°. The best 5-figure table is *Tafeln für numerisches Rechnen mit Maschinen* by O. Lohse. First edition, 1909; second edition, 1935. This gives six functions for every hundredth of a degree. No recent available table, however, gives more than seven figures, apart from the one now under review.* *Trigonometria Britannica* thus still leads the field after 300 years.†

Buckingham's table therefore occupies an important position, in spite of its rather too many faults. An added importance arises from the adoption by the Mathematical Tables Committee of the British Association of an eight-decimal standard for many of its tables, and from the practice of tabulating oscillatory functions in the form $F(x)\sin\theta(x)$ which is being used in the forthcoming tables of the Airy Integral (Watson, in his *Treatise on the Theory of Bessel Functions*, 1922, pp. 666, 714, gives $J_0(x)$, $J_1(x)$, $J_{1/3}(x)$ in this form but with sexagesimal subdivision, making interpolation very troublesome). Again, the tabulation of complex functions is frequently simplified by giving modulus and phase. A most important advantage enjoyed by Buckingham's table at present is, however, that it is American and can be obtained in war-time.

For these many reasons the " discovery " of Buckingham's table by Comrie was hailed with considerable delight by the writer, and a careful examination of the book has been made, mainly by S. Johnston, the results being collected and analysed by Comrie.

The main table, pp. 8–97, gives 8-decimal values of the sine, cosine and tangent, and 8-figure values of the cotangent; in each case the first difference is also given. The arrangement is semi-quadrantal, that is, all four functions appear on a single page, and the arguments run from 0° to 45°; each opening covers one degree. The user is left to decide on the adequacy of linear inter-

* L. W. Pollak, in his *Handweiser zur harmonischen analyse*, Prague, 1929, p. 38, refers to a table *Zehnstellige Tafel der Sinus, Cosinus und Tangenten für die dezimale Teilung des Nonagesimalgrades*, calculated by E. Engel, which gives 10 decimal natural values. This is stated to have been published in 1920 by the 'Generaldirektion des Grundsteuerkatasters (Österr. Triangulierungs- und Kalkulbureau)'. Further details of this table are unknown, except that it provides tables of proportional parts, although second difference interpolation is apparently necessary.

† Logarithmic values of trigonometrical functions are better represented in spite of their diminished importance in these days of calculating machines. This is due to three works by Peters: *Zehnstellige Logarithmentafel*. Band II. 1919. *Siebenstellige Logarithmen der Trigonometrischen Funktionen*. . . . 1921. *Sechsstellige Logarithmen der Trigonometrischen Funktionen*. . . . 1921. These give respectively 10, 7, and 6 decimal values of four functions for every thousandth of a degree. Bremiker's *Logarithmisch-Trigonometrische Tafeln mit fünf Decimalstellen*, 1872, is also worthy of mention on account of its influence on the revival of the degree-and-decimal subdivision.

polation, and no extra provision is made when this is inadequate, namely, for the cotangent of angles less than 30°, and for tangents between 28° and 45°.

Ten errors in this table exceed a unit of the final decimal, as follows :

Page	Argument	Function	For	Read	Page	Argument	Function	For	Read
12	2·27	cot	227	224	21	6·74	cot	093	095
	2·29	cot	670	667	22	7·46	cos	576	574
14	3·24	cot	099	029	26	9·27	cot	493	495
15	3·51	cot	153	158		9·28	cot	305	307
20	6·40	tan	11226..	11216..	47	19·76	cos	704	702

In every case, except at tan 6°·40, the corresponding differences are also in error, showing clearly that they have not been used to check the table. Besides these errors over 460 cases of a discrepancy of a unit in the final decimal were found on comparison of the table with every 36th value in Peters' *Achtstellige Tafel des trigonometrischen Funktionen für jede Sexagesimalsekunde des Quadranten*. Berlin, 1939. In view of Peters' known high standard of accuracy, and the fact that the calculation of this table was done in collaboration with Comrie, these discrepancies are almost certainly all errors in Buckingham's table—although admittedly very minor ones.

The conclusion reached is that, subject to the application of the corrections listed, the table under review does fill the 8-figure gap adequately, being of 8-figure accuracy for all practical purposes.

The remaining tables—nearly half the book—are also of some interest. The second table is a one-page table for converting minutes into decimals of a degree ; the argument proceeds by quarter-minutes up to one degree.

The third table, pp. 100–129, gives inv ϕ, by which is meant $\tan \phi^\circ - \pi\phi/180$ or $\tan \theta - \theta$, if θ is the radian measure of ϕ. This definition is nowhere given in the volume. The interval throughout is 0°·01 ; 12 decimals are given up to 0°·50, then 10 to 1°·00, 8 to 37° and finally 7 to 60°, the end of the table. The table was examined, but not with a view to finding errors so small as a unit in the final decimal. Two larger errors were found :

inv 9°·15, p. 104. For 6160 read 7160
inv 13°·01, p. 106. For 468 read 470

The fourth table, pp. 132–146, gives the circular measure to eight decimals for every 0°·01 up to 45°. Comrie comments : " This table has been computed by taking multiples of 0·01745 329 (exactly) instead of 0·01745 32925 ... up to 17°·96. The values for 17°·97, 17°·98 and 17°·99 have been arbitrarily increased by 1, 2 and 3 units in the 8th decimal. The value for 18°·00 and all subsequent values have been increased uniformly by 4 units. This still leaves 18° in defect by 1 unit, and the defect steadily increases to 7 units at 45°."

The fifth table, pp. 148–169, is of some interest. It is called " Brocot's Tables of Gear Ratios ", and gives the values to 8 decimals of all fractions N/D, where N and D are mutually prime integers, $N < D \leqslant 120$. Brocot's original table, which appeared in 1862, extended only to $D \leqslant 100$, and gives 11 decimals. It was anticipated in 1818 by Goodwyn's table, also with the limit 100 for D, but with $N \leqslant \frac{1}{2}D$; Goodwyn gives material for obtaining complete recurring decimals. (The values for $N > \frac{1}{2}D$ are of subsidiary importance, since $(D-N)/D$ is so easily formed.) *Machinery's Handbook* gives a table, also called Brocot's, with $D \leqslant 60$, and it is worthy of note that of the 27 errors and omissions in Buckingham's table, all but one have

the denominator exceeding 60. Excluding unit errors in the final digit of decimal values, the errors occur at the following arguments :

Omissions - - -	3/86, 3/70, 46/103, 41/79.
Argument errors - -	41/100, 95/109, 44/47.
Decimal value errors -	3/106, 9/67, 22/101, 38/107, 46/117, 39/79, 67/105, 59/89, 62/79, 61/73, 98/117, 107/112.

The final table, pp. 172–183, gives factors of numbers from 1 to 6009. Two errors have been found : $5183 = 71 . 73$ and $5461 = 43 . 127$ are given as primes.

The book is " Reproduced by the photo-offset process directly from the original manuscript . . . " and the result is a clear legible table. Considerable care has been expended on the preparation of suitable forms, and the values have been very carefully written in. The writer prefers a well-printed table, but has to admit that the table under review seems unlikely to cause eyestrain or other difficulty.

A one-page preface gives no information about the preparation or source of the tables, nor about the accuracy claimed ; all or any of this information would be of interest to users of the tables. It may be, of course, that a fuller explanation is reserved for Section Two (which the writer has not seen, and which is apparently a purely engineering work), but there is no indication of this in the preface.

One further comment must be made. Four copies of the table have been examined ; one is clearly a reprint, as it differs in several respects from the others. Mention need only be made here of the correction of certain errors—the preface does not mention this. One error, tan 6°·40 on p. 20, seems to have been corrected by hand after reprinting. The others, arguments 95/109 and 44/47 on pp. 167, 168 and $5183 = 71·73$ on p. 182, appear to have been corrected before reprinting. The method of correction used on pp. 167 and 168 does not appeal to the writer.

To sum up, Buckingham's main table adequately fills a very important gap, and the remaining tables may also be found useful. It is necessary, however, before using the tables to correct the major errors listed above. The table of circular measure should not be used for accuracy above 7 decimals ; in view of the remarks on calculating machines in the preface, it is difficult to see why this particular table was included. J. C. P. MILLER.

LOG π.

Decimal Hoppus Tables. By S. E. WILSON. Pp. 60. 6s. 1941. (Scientific Computing Service)

Few outside the trade will have heard of HOPPUS. The uninitiated may assume it to be a drink, a disease, or a colloquial synonym for oast-house. The word was unknown to me until I received for review a small book with the intriguing title of

DECIMAL HOPPUS TABLES.

Evidently Hoppus (whatever it was) lent itself to mathematical treatment, and was important enough for the attention of the Scientific Computing Service, who are the book's publishers. The preface, however, quickly shattered an illusion that would not have been created if the title had read, as it should,

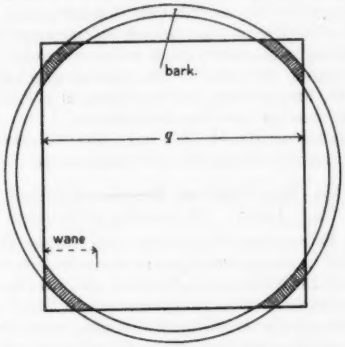
DECIMAL HOPPUS' TABLES,

from which it would have been clear that Hoppus was a proper name (cf. Bumpus, Cardus).

Timber merchants naturally need a method of computing the quantity of wood in a felled tree or log. They are interested, not in what they would call

the " mathematically correct " volume, but in the amount of useful timber that can be got out. Ideally a felled tree is a truncated elliptical cone, but this is rarely realised even approximately in practice. The geometrical " volume " of the tree would be sufficiently accurately found from the overall length and the section at mid-length, although the measurement of major and minor diameters at mid-length involves the use of sets of large and clumsy callipers.

Three centuries ago, or thereabouts, it was noticed that the width of a log roughly squared with axe and adze, and with wane at the corners, was practically equal to a quarter of the girth of the original round log. The square of this mid-length quarter-girth, multiplied by the overall length, gave a sufficiently good indication of the useful timber content of the tree. The quarter-



girth (q) in inches is obtained easily by means of a direct-reading tape measure, and if the length of the trunk is l feet the " true " volume of a cylindrical log is $q^2l/36\pi$ cubic feet. The British Customs uses $q^2l/113$, since $36\pi \simeq 113$ (and not on account of $\pi \simeq 355/113$). The roughly squared log has a volume of $q^2l/144$ cubic feet if the side of the square is taken as q, and it is this formula which is used in the so-called Hoppus' system. One cubic foot of " content " will correspond to $4/\pi$ or 144/113 or 1·273 cubic feet of " volume ". Confusion is avoided by using the term " Hoppus' foot " for the unit of volume computed from $q^2l/144$.

Edward Hoppus was a surveyor to the Corporation of the London Assurance, and died in 1739. His *Practical Measuring made Easy to the Meanest Capacity by a New Set of Tables* was an improvement on Keay's *Practical Measurer's Pocket Companion* and on Darling's *Carpenter's Rule made Easy*, and so the system which he did not invent has come to be known by his name. An extract from his preface illustrates the differences between the units in which the rival tables are expressed : " . . . this Method is more *intelligible*, and *better suited* to *ordinary Capacities* than either of theirs. These *Parts* of an Inch are *easily* cast up, and readily understood ; for here you have nothing more to do than *to carry* 1 to the *Inches* for every 12 in the *Parts*, and 1 to the *Feet* for every 12 in the *Inches* : Whereas by *Keay's* Method you must carry 1 to the *Quarters* for every 432 in the *Inches*, and 1 to the *Feet* for every 4 in the *Quarters* ; and by *Darling's* Method you must carry 1 to the *Feet* for every 10000 in the *Parts*. Both which Numbers of 432 *Inches* and 10000 *Parts of a Foot* are very *difficult* to conceive, and consequently very *troublesome* and *perplexing* to every Man that is not versed in *Decimal Arithmetick* and

Mensuration." After two hundred years use, Hoppus' duodecimals have given way to decimals, in which " ordinary capacities " are better versed.

This mid-length-quarter-girth system has been criticised as " mathematically inaccurate ", and that the more sharply tapering the tree the smaller the calculated volume compared with the true volume. But the system aims at giving the *useful* timber-content, and there is a smaller proportion of useful timber in a sharply tapering log. The reason for the use of quarter-girth instead of whole-girth is obvious when one considers how it widens the scope of the tables. For round logs the tables give the volume in Hoppus' feet, for roughly squared logs the approximate timber-content in cubic feet, and for square prisms the exact volume in cubic feet. The system involves easy and few measurements, needs pocket tapes only, and a simple set of tables. Even without tables the only divisor needed is 12, and π is avoided. Could an International Commission on Timber-Measurement have produced anything as simple and effective as this time-honoured system?

The lay-out is clean and clear, and relies entirely on spacing, with no irritating " lines to guide the eye ". A typical page gives the contents of, say, a log up to 50 ft. long whose quarter-girth is 23″, 23¼″, 23½″, 23¾″. The book is a convenient size and is thumb-indexed.

I have to thank my friend Mr. W. R. Crow, the well-known timber merchant, for much information about Hoppus and his system. A. P. R.

An Inquiry into the Prediction of Secondary-School Success. By W. G. EMMETT. Pp. 58. 2s. 1942. (University of London Press)

The West Riding Education Committee and Moray House, Edinburgh, are both keenly interested in the procedures for the selection of pupils for secondary schools, and their joint efforts have given us several reports on the problems. In 1938 the N.U.T. conducted an investigation in the West Riding, and some useful information about the working of this local education authority is given in the Report, *Special Place Examinations* (University of London Press, 1941), as well as in *The Selection of Children for Secondary Education*, by Mrs. Thomson Davies and Mrs. Jones, the Chief Examiners of the West Riding. A favourite topic has been the attempt to compare the relative efficiency of an examination in arithmetic and English with one using intelligence tests. In this, writers have followed some suggestions of Valentine in his *The Reliability of Examinations* (University of London Press, 1932). Ferguson in *The Reliability of Mental Tests* (University of London Press, 1941) has submitted some work based on the West Riding 1938 experiment to demonstrate the general efficiency of the Moray House Tests. Now we have this booklet by another member of the Moray House staff. Mr. Emmett is unfortunately now " away on national service ", and his colleagues of Professor Godfrey Thomson's staff have seen his book through the press. Presumably, Mr. Emmett is responsible for the general research here reported. He looks at the after records of 765 children (471 boys, 294 girls, *vide* p. 18) who entered one or other of seven West Riding secondary schools in 1933 and 1934. The main selection then was by an $A+E$ examination, with an age allowance computed on the total, and an Intelligence Test used " for the border line group " (p. 16 : there is no statement of the width of the border line, nor of the method of using the intelligence test here). Secondary School " success " was an assessment made in each of the secondary schools concerned, ranking the children in an order of merit. The conclusion reached is that the Intelligence Test is significantly superior for prognostic purposes to an examination using arithmetic or English. The reviewer submits that the writer's case is not proved.

(1) It is true that the writer used a correction for the shrinkage in correlations due to selection. This, as has been pointed out before (*e.g. Annals*

of Eugenics, 1936, pp. 65–85), is necessary, and most, if not all, the writers in this field have failed to avoid the fallacies involved (*e.g.* Valentine, Amos): many have not even realised the difficulties. But Emmett's method is still unsound, for two reasons.

(*a*) The formula used is, apparently, that given by Karl Pearson in his paper "On the Influence of Natural Selection on the Variability and Correlation of Organs" (*Phil. Trans.* A, 200, pp. 1–66, 1902). But Pearson's selection is entirely different from the selection that we have here. In Pearson's paper (p. 18) he makes clear his type of selection—an attempt to get a measure *round about* a pre-assigned value. In selection for secondary schools, we have a sharp guillotining—above a definite value, in; below a definite value, out. And the border-line procedure only makes a slight haziness round the cut, and would certainly not make the curtailed distribution a normal one, as Pearson wants (*loc. cit.*, p. 2).

(*b*) The constants of the three variables used (*A*, *E* and *I*) in the selected and unselected populations used for substitution in Pearson's formula are not comparable. For the unselected population Mr. Emmett uses (p. 31) the scores made in the 1938 experiment. But, as the Chief Examiners point out (*e.g.*, p. 37, of *Special Place Examinations*), in this the experiment was made by using the usual type of *A* and *E* question papers though the field of candidates was not the usual one of children already selected by a preliminary test to have in the main an I.Q. lying between 100 and 129, so that their question papers in consequence had badly distorted means and s.ds.

(2) The age question is not satisfactorily dealt with.

(*a*) The correlations of secondary school success are with I.Q.'s, that is, with the Intelligence Test score corrected for age, and with the *A* and *E* raw scores, uncorrected for age. Emmett claims (p. 23) that the error thus introduced is small, but his reason for so doing is quite wrong. Of course, in any one class, the correlation between age and score is small, at least if the secondary school headmaster does his job and gets his classes as homogeneous as possible. The correlation may even be negative (vide *Forum of Education*, 1925, pp. 24–31). But this does not mean that for any one individual there is not, perhaps, an enormous increment of score in one year in a test such as that used in a secondary school entrance examination.

(*b*) It is well known that teachers' orders of merit, or marks, the criteria of the Pearson formula, do not usually (Emmett, p. 23) and cannot actually (*Special Place Examinations*, p. 49) allow for age (cf. *Brit. Journal of Educ. Psych.* 1933, pp. 269–290). Possibly it is the asymptotic approach of the teachers' assessment to a final ranking independent of age that is the cause of the increments (if real, p. 34) from 2-year to 3-year in the correlations of each measure with the criteria.

The four constants therefore of the Pearson formula are not on truly comparable bases, and hence it seems doubtful, even if the use of this formula were justified, whether the numerical results obtained could be interpreted as the writer interprets them in this booklet.

(3) It is further possible that the lumping together of boys and girls (p. 18) has introduced a masking of the real effects. The reactions of boys and girls both to the different parts of an entrance examination and to exposure to a secondary school environment are so different that the sexes should always be separated for statistical analysis if possible (vide *B.J.E.P. loc. cit.*).

(4) The effects due to selection having been recognised, we may remember the rule "the other test tends to seem the better" (*Annals of Eugenics, loc cit.*). As different examinations use different weights among the selecting question papers, either deliberately or as a result of the type of questions or of the schedule of marking, and the intensity of selection differs widely—even

among one local education authority, (vide *Special Place Examinations*, p. v): is this the explanation of the differences in the ranges of r for schools 2 and 3, 1933, Junior, of Table 4, Emmett?—it does not seem a sound conclusion that the relative prognostic values of the various examinations (p. 35) are fairly steady.

(5) The writer draws a number of conclusions (pp. 37, 38, 42, 44) about the relative superiority of the Intelligence Test and the A and E, either separately or as members of a battery. All the results here depend on the correlations in sample and population as a whole, and we have already indicated that there are some unjustified assumptions in the use of the values employed.

(6) Table 3 (p. 18) gives means and s.ds. of I.Q.'s for each school for each type of entrant (senior and junior, 1933 and 1934). It is difficult to appreciate why the seniors of 1934 had a higher I.Q. than the juniors of 1933, for they were of the same age group, and as a rule a two-year age range in a secondary school entrance examination is creamed in the first year, that is, the I.Q. of the juniors should be higher. The reviewer attempted to use the figures to analyse the matter further, but failed to get the statistics of each group as given in the table from the separate school values: it is not clear why Sheppard's corrections are used between the computation of the variances of schools and the computation of the variances of types.

We have given rather full attention to this little pamphlet, for it is a treat to meet an investigation that is really a serious attempt to deal with a problem that so many of the critics—and supporters—of examinations fail even to realise exists. The present inquiry is another milestone on the way to a proper understanding of the technique of examinations, and the principle of scrutinising examinations, and the book should be seen by all interested in these problems. FRANK SANDON.

The Fundamental Principles of Mathematical Statistics, with special reference to the requirements of Actuaries and Vital Statisticians and an outline of a course in Graduation. By HUGH H. WOLFENDEN. Pp. xv, 379. 5§. 1942. (Macmillan, Canada, Ltd.)

This book is sponsored by the Actuarial Society of America (New York), working through its Committee on Actuarial Studies, and is copyrighted by them. The author is a Canadian lecturer in mathematical statistics to actuarial students, and the book was printed, at the University of Toronto Press, in pursuance of the educational policy of the A.S.A. As the author explains, he has attempted to assemble and coordinate those portions of the theory and application of mathematical statistics that are needed in the highly specialised studies and practical work of the actuary. The book accordingly gives more prominence than is usual in English statistical textbooks to the work of other actuarial statisticians and to the Scandinavian school, though the English workers, *e.g.* Edgeworth, Karl and Egon Pearson, Student, and R. A. Fisher, are on the whole fairly dealt with as far as their work is relevant. The author, with the actuary's distrust of assumptions of homogeneity, makes a special point of tests of homogeneity, and so Lexis is more fully considered than is common. The writer says that he assumes only the elements of probability and of the calculus, but actually there are large parts that would require a good working knowledge of finite differences. In this connection we may note that the sections on graduation are exceptionally fully done. The conflicting demands of the tests of "fit" and of "smoothness" are specially emphasised. The frequency of change of sign rule does not commonly appear in statistical books, and the warning that the good fitting curve will not necessarily give a better prediction is well put.

The book is really an outline of a course, but it would present, I think, great difficulties to any unaided student. It is in four parts. The body of the treatment gives a condensed presentation of the elements of the subject. This takes 148 pages: it is not stated how many hours (or terms or years) the student would have available to master it all in the author's course of lectures. This part is followed by three others dealing respectively with historical notes—to give background—technical analysis—mathematical glosses and interpretation—and a section on applications and examples. This treatment results in many awkward cross-references: the order in which the parts should be read is not clear, as the curious tense usages (*e.g.* on pages 29 and 132) indicate. There are two bibliographies—works of historical interest and those of present value. The apparent distinction intended by the writer breaks down, in the opinion of the reviewer, in practice. Books appear in the P section and their reviews in the H section; a number of modern papers are in the H section that might, it seems, as well have appeared in the other. Together the bibliographies are very useful, though their value would have been increased had references been given more fully. Often it is difficult to know if a title is that of a book or paper, especially as publisher and journal are sometimes omitted. One edition, not always the latest, is listed in cases where volumes have run into more than one edition. The lists would have probably been of greater value if there were collected together references to those publications which contain tables, nomograms, etc., likely to be needed by the actuary. In the courses dealing with curve-fitting we are given useful references to data that have been fitted with curves of various types.

Although the idea of the book appears to be quite definitely to send one to the sources and to standard works, yet we find reproduced here a number of old friends—Arne Fisher's urn drawing illustration for the Bernoulli, Poisson, and Lexis sampling (this is in the mathematics section), the Prussians whom Bortkiewicz found kicked to death by horses (this is in the history and the applications parts), the soporific efficiency of dextro- and laevo-hyoscyamine hydrobromide (" applications " section), and Weldon's 26306 throws of 12 dice (" applications ": even if the test gives ·3376986 as a more likely value of p than ·333 ..., it surely is not correct to say " that there is no reason to doubt the hypothesis that the probability for each die of throwing 5 or 6 was actually the figure ·3376986 "). The author seems to have information not available to the distinguished actuary who reviewed *Facsimiles of Two Papers by Bayes* in the *Math. Gazette*, 1941, July, XXV, 265 (p. 177), for (p. 165) he describes Bayes as " a dissenting minister, Thomas Bayes, F.R.S."

There are a few diagrams, chiefly of the Pearson curves; it is a pity that none of the U curves are depicted. The perspective views of smooth curves in space should not give the sharp points of $C1$ (p. 201) while there seems no justification for interchanging the axes of the ogive (p. 193) to get vertical asymptotes (see figure, p. 199) instead of the usual horizontal ones.

The book is well printed, both in the straight text and in the mathematical parts, and we have found very few misprints. The author, following actuarial practice, uses p and q for his two probabilities with the usual idea that q is a rate of mortality, and p of survival (and thus of order unity). His binomial expansions are thus of the form $(q+p)^n$, but the letters have been interchanged on p. 270.

In all, we have here a sound and useful volume, and the student who is fortunate enough to " plan his reading systematically " with the help of this book, to have the guidance of a good teacher, and to have access to a library containing the works in the bibliography, will have a real appreciation of statistical method as applicable to actuarial and allied problems.

FRANK SANDON.

Mathematics. Its Magic and Mastery. By A. BAKST. Pp. xiv, 789. 21s. 1941. (Chapman and Hall)

This is a book very suitable for a school library, or, except for its price, as a mathematical prize or present for an intelligent youngster. On the whole its claim is justified : " This book is designed to make mathematics interesting . . . the book is sufficiently complete so as to give a broad picture of mathematical fundamentals." Whether it will appeal to the adult reader who feels inclined to rub up or supplement his half-forgotten schooling will depend to a certain extent on how the reader reacts to the headlines and comic-strip drawings which are superimposed on a comparatively sober account of mathematics.

The book does not need for its full appreciation such a wide range of outside knowledge as does *Mathematics for the Million*. There is no savage digression on the Athanasian Creed, and the author is not striving to prove a thesis as Professor Hogben is concerned to prove that mathematics divorced from contemporary life and affairs is nothing.

A good deal of arithmetic, algebra, geometry and trigonometry is covered and a little mechanics. A brief synopsis may give some idea of the contents.

The first eleven chapters deal with arithmetic and the theory of numbers—2 and 3 as possible bases instead of 10 come in chapter 2. Of these chapters, which contain much that is interesting, chapters 4 and 5 on large and small numbers or rather " number giants and number pygmies ", chapter 6 on codes and ciphers, and chapter 8 on calculating devices are perhaps the most successful.

Algebra starts at chapter 12. Chapter 14 with the title " The grammar of algebra " consists of a series of problems with an element of humour in them (for example, " beware the math.-minded mother-in-law ", " the erring speed-cop ") and has none of the dullness one is apt to associate with " grammar ". Chapter 16 on instalment buying and chapter 17 on chain letters and the G.P. are interesting ones. " The banker's number—Jack of all trades " introduces e, primarily through compound interest and then as involved in formulae for growth and decay. Chapter 20, " How to have fun with Lady Luck ", deals with probability.

In the geometrical sections much play is made with the idea of dimensions and Flatland, not without a suitable reference to " E. Abbott's small but fascinating book ". The equation of a curve becomes the " passport " which permits travel to the inhabitants of a point world. Chapter 24, which deals with such passports, seems likely to prove decidedly difficult to the untaught reader.

The headlines mentioned above seem less successful in the geometry section than elsewhere. In a sober account of diameters and chords of a circle the remark is inserted that " the chord is the diameter's first cousin ", mainly perhaps to justify the headline " The circle family : meet cousin chord ". The circle itself appears first under " Life in a merry-go-round ". As a chapter heading for " the shape of things " there is a thumbnail sketch of a beauty contest. Two chapters have the vigorous titles " Man's servant, the triangle " and " The triangle, man's master ", but it is hard to see any reason why these two titles should not have been interchanged, unless the suggestion is that by the introduction of " The trigonometric trio : sin cos and tan " in the latter chapter, man has been reduced to subjection ; perhaps this is because the sine is " the ratio that does everything ".

Solid geometry is given the new title of " Cork-screw geometry ", and contains such things as the spider-and-fly problem and how to find the geodesic on a pyramid.

With chapter 35, "Mathematics, interpreter of the universe", we reach kinematics and dynamics; the use of such formulae as $s=\frac{1}{2}at^2$ and $g=v^2/r$.

In chapter 36, "The firing squad and mathematics", and chapter 37, "Of maths. and magic", are packed a good many miscellaneous items of interest, for instance an account of "looping the loop", and these bring the book to an end except for a voluminous appendix.

The appendix gives 43 pp. to a summary of the rules of algebra, geometry and trigonometry, and 40 pp. to tables and their use, answers with hints for solution and an index. The tables given are logarithms, squares and square roots, and the (natural) sine, cosine and tangent tables.

It will be seen that the book is a large one, and thus in spite of the wide extent of ground covered the explanations are entirely unhurried. The advertisement on the loose cover says "The material is simply developed. No proofs of any kind are used in the unfolding of the mathematical processes and properties". This second sentence is an exaggeration. There is a good deal of simple proof; for example, the usual proof of the extension of Pythagoras' theorem is given (p. 497) though the original theorem is assumed without proof.

The book has been lying on my table for a couple of months. It improves on acquaintance. Each time I have dipped into it I have noticed its excellences more and its feeblenesses less.

C. O. TUCKEY.

Tutorial Algebra. II. Advanced Course. By W. BRIGGS and G. H. BRYAN. 5th edition. Revised and rewritten by G. WALKER. Pp. viii, 599. 12s. 6d. 1942. (University Tutorial Press)

It is necessary to recall the words of Dr. Heywood, said at the Annual General Meeting of the Mathematical Association on 6th January, 1925 (see *Gazette*, XII, p. 324): "The teacher is given a number of students and is expected to prepare them for degrees, which means for examinations. If the students do not pass their examinations they will be dissatisfied, and the College authorities will be dissatisfied with the teacher. The teacher, therefore, to save his own skin becomes a coach. He is none the less a coach because he and his college may pretend he is not. The textbooks used may or may not be written as cram books. But it is certain that if a book is not suitable for examination purposes it will not be used."

It is proposed to examine this book in the light of these remarks, and we may at once gain support for Dr. Heywood's contention by quoting a short extract from the Preface: "Volume II covers algebra necessary for the B.A. and B.Sc. General examinations of the University of London, and would serve as an introduction to the Honours examinations, since the ground covered extends beyond the scope of the General Degree. It is also hoped that this volume will be of material assistance to students who are reading for University Scholarship examinations as well as to those studying mathematics for the Natural Science and Mathematical Tripos examinations of the University of Cambridge."

This is a big book, well printed, with full answers and a good index; and we may conveniently use the author's own account of its contents: "Chapters I and II deal with limits, convergence and continuous functions, and reference is made to the O and o notations, while Chapter III considers the Binomial Theorem for a rational index. After a discussion of uniform convergence in Chapter IV, the properties of the exponential and logarithmic series are developed in Chapter V. This is followed by a chapter on the determination of functions from empirical data. Properties of complex numbers and series are considered in Chapter VIII. Chapter IX contains a general discussion of partial fractions and recurring series. Methods of summation occupy

Chapter X. The two succeeding chapters, XI and XII, present the fundamental properties of determinants and elimination. The two final chapters are devoted to general properties in the theory of equations, and include, in particular, methods of approximation to the roots of an equation."

The work on convergency is very thoroughly done, and it suits the author's sequence to establish the existence of

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

by means of Tannery's theorem. Still it seems a pity not to have given as well the proof depending on Weierstrass' inequalities, as found, for example, in Hobson's *Trigonometry*.

Strong exception must be taken to the following sentences, to be found at the top of p. 274 : " As another example consider amp z. This is a multiple-valued function, but is made *single-valued* by considering the *principal value* of the angle. This is equivalent to restricting ourselves to the consideration of a single branch." The italics are as printed in the book, and it is not easy to disentangle the confusion of thought contained in these sentences. Suffice it to say that they ought never to have been written.

It is not at all clear why the author, in the last chapter, changes his mind about notation. In the rest of the volume suffix notation is consistently used, but now we find

$$\text{" } ax^3+3bx^2+3cx+d=0\text{ "},$$

and " assume that

$$a(ax^4+4bx^3+6cx^2+4dx+e)=(ax^2+2bx+c+2a\theta)^2-(2px+q)^2\text{ "}.$$

This is shockingly bad teaching : there is no need to assume anything. The work should be presented as follows :

$$a_0(a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4)$$
$$\equiv(a_0x^2+2a_1x+\lambda)^2+[2x^2(3a_0a_2-2a_1^2-a_0\lambda)+4x(a_0a_3-a_1\lambda)+(a_0a_4-\lambda^2)].$$

Making the second term a perfect square, we get

$$(3a_0a_2-2a_1^2-a_0\lambda)(a_0a_4-\lambda^2)-2(a_0a_3-a_1\lambda)^2=0$$

or
$$\lambda^3-3a_2\lambda^2+(4a_1a_3-a_0a_4)\lambda+a_4(3a_0a_2-2a_1^2)-2a_0a_3^2=0.$$

Writing $\lambda=a_2+\mu$, which is forced on us, we get

$$\mu^3-I\mu+2J=0,$$

and if we like we can then put $\mu=2a_0\theta$.

In spite of the foregoing criticisms it is not to be concluded that a student, unassisted by oral instruction, will go far wrong in using this book. It is possible too that the author's somewhat pontifical style (as on p. 274) may give him a feeling of security which may tide him over his examination, and which he will not realise as false until afterwards—perhaps not until long, long afterwards. And so we come back to Dr. Heywood again. N. M. G.

First Steps in Astronomy without a Telescope. By P. F. BURNS. Pp. ix, 214. 5s. (Ginn)

The book opens with a quotation from the immortal " Alice "—an excellent opening to an excellent opening chapter—but had Alice penetrated as far as page 185 she might have been heard remarking " Curiouser and curiouser ", for on that page we find—can it be a Square Degree? It takes the form : " Jupiter was $(7^\circ\times15^\circ)+8\frac{1}{2}^\circ\ldots$ " and might well have been overlooked but for the fact that similar sums occur on pages 186, 187, 188 and 189!

I am completely in sympathy with what I take to be the aim of this book, namely the stimulation of elementary mathematics by supplying interest value—but the author has fallen into the almost inevitable trap of making misleading statements in his attempt to arouse interest. For example, he says (p. 25) : " . . . the navigator is able to find the longitude " by observing " the reading of his chronometer . . . when the sun is due south ; at that moment the ship's *local time* (his italics) is noon. . . . The ship's time is then . . . compared with Greenwich *Mean* Time (my italics!). . . ." Would that the process were as simple as that. Such an observation is not practically possible : even if it were, the incorrect use of " mean " could have been avoided by referring to " Sun Time " or even, at this stage, suppressing the word altogether. Incidentally, I think it is a great pity that the only reference to Mean and Apparent Time occurs in a note to one of the examples : it is so easy to tell the young handyman how to make a simple form of sundial—and the main difficulty is then over.

In spite of a number of blemishes, of which I will mention a few more, the book is well worth its modest cost, and I can recommend it to schoolmasters, many of whom should (and I hope their pupils will) learn a lot from it. It is pleasant to look at, the presentation is simple and the figures good—but :

Why (p. 9) is " angular distance measured at the centre of the earth "? Surely the whole conception of angular distance depends on the fact that it is an arc rather than an angle which is visualised. The looseness with which this most useful phrase is treated is also evident in the definition of longitude (p. 22) as the " angular distance east or west of an agreed place such as Greenwich "—a statement which will not bear analysis.

Of what value are questions of the following type, of which there are many? " How long does Aldebaran remain above the horizon . . . ? ", the answer being obtained by taking the difference between the times of rising and setting as given in a table of statistics.

Is it really desirable to define the cosine of an angle (and in a somewhat specialised form) or to explain the use of cosines and, later, tangents and provide a 3-figure table of their values? Why not let the book achieve its ostensible object—a very laudable one—by showing the necessity for trigonometry?

Would not a little more consistency (even if it *is* the bane of little minds) be helpful in the labelling of figures? There is, for example, one figure in which one might not unreasonably expect to find the points of the compass : in this figure N, S, E and W are clearly shown—what a surprise, therefore, when " W " turns out to be Wellington (N.Z.) and " E " is the equator which, in a later figure, becomes " Q " because " E " is then wanted for Edinburgh! Poor " N ", the north pole on one page is, on the previous page, New York.

The book deserves a large sale if only for the fact that there is an excellent explanation of why the Australians do *not* stand on their heads, and also for pointing out that " in most of our English schools it is not possible to *set the globe* " (his italics—and mine) so as to show the positions of the poles and equator relative to the observer who, strange as it may seem, thinks he is standing upright on his feet. The habit of drawing pictures of the heavens with the poles at the " top " and " bottom " is, in my opinion, a most reprehensible offence. May I point out, in this connexion, that an umbrella with the ferrule pointing to the Pole Star gives, when rotated, an excellent illustration of the apparent motion of the heavens.

I much appreciated, and could have done with more of, the little excerpts from mythology and the introduction of the human element.

One puzzle remains : the author explains how the order of the names of the days of our week can be derived from " seven wanderers " by arranging

them in a certain astronomical order in a circle and then counting a succession of 25's—but *why* 25? The same result could have been achieved by counting 4—or even 3 if the count started from the next in line. No hint of the mysterious potency of 25 is given: I am left wondering. G. A. C.

Engineering Mechanics. By B. B. Low. Pp. viii, 252. 12s. 6d. 1942. (Longmans)

This book is intended as a companion to D. A. Low's *Applied Mechanics*, and consists of disconnected chapters or groups of chapters dealing in greater detail with some of the more fundamental aspects of mechanics. The topics are: Kinematics, treated both analytically and graphically (Chapters I–IV, to which may be annexed VIII on "The Analysis of Cams"); Dynamics, of particles and the motion of a rigid body in two dimensions (Chapters V and IX); Vibrations, including those of beams and the whirling of shafts (Chapters VII and X); Dimensions and Dynamical similarity (Chapter VI); Deflection of Beams (Chapter XI).

It cannot be doubted that the book will be welcomed by those to whom it is primarily addressed—engineering students. Explanation is never stinted, whether in the text or in the numerous worked examples; dull indeed must be the student who complains that he is unable to follow because of the omission of steps in the argument or calculations. Many exercises are provided and answers to all are given.

Teachers of mechanics—not only to engineering students, and at various levels—will find much useful material in the book. The discipline of velocity and acceleration diagrams, for instance, might find a greater place in the schools, and the *use* which is made of geometrical facts and constructions should commend it to the secondary school teacher. The exercises, too, contain many "real" "live" problems which could with advantage leaven the teaching of mechanics.

A few detailed comments, favourable and otherwise, occur at this level. In the chapter on instantaneous centre, it would seem desirable to include the fundamental proposition, that *any* plane displacement of a rigid body can be effected by a rotation. It is a pleasure to see Newton's third law completely stated. The mass-weight (or units) difficulty is attacked at some length, but (I fear) not completely overcome, although in the chapter on "Dimensions" the engineering system (fundamental units length, time, and *force*) is given prominence at least equal to that given to the more usual LTM system. If only the engineers would use lb.-wt. when they mean force, and reserve lb. for mass, their students would be greatly helped. On p. 182, after defining ρ as *density*, our author uses ρ/g. A curious omission in the chapter on S.H.M. is the formula connecting velocity with displacement, usually given in the form $v^2 = \omega^2(a^2 - x^2)$. Motion of a rigid body is treated (consistently, clearly, and well) in terms of the equivalence between the external and "effective" force systems. (But the "proof" of the principle of energy for general plane motion is incomplete, since no independent demonstration of the formulae for kinetic energy, or work done, is given.) In the case of the beam with clamped ends and a central load, only three of the end conditions are used, and the satisfaction of the fourth is not verified.

At a higher level of criticism, however, one cannot but be irritated by a number of points of detail which indicate carelessness, haste, or a kind of mental blindness. Engineers are often curiously insensitive to precision in the use of words. For instance, after emphasising the usual distinction between "velocity" and "speed", these terms are *not* used with discrimination. "Instant" is used to denote a short interval of time. Terms are "neglected" when they have indeed been considered, and also where (as in

Macaulay's method) their *omission* is part of the technique. Frequently the word " component " could (often should) be inserted. " Equation of motion " is used with several significances.

Apart from the mass-weight confusion, the author's use of specific units is erratic. For instance we have " a beam of length l, loaded with w lb./ft. . . . " and no units are specified for E and I, for instance. For literal quantities units are unnecessary—and probably undesirable—but surely they should be specified either for all quantities or for none.

The velocity-speed case is only one instance of many in which the author seems not to have the courage of his convictions. On p. 72 we read " . . . pound-weight-feet, or more briefly, although strictly incorrectly, pound-feet ". When it has been admitted that a beam section can have neither a centre of *gravity* nor a moment of *inertia*, the correct terms centroid and second moment occur only occasionally, and seemingly by inadvertence.

Nor is there any consistency in the set-up of the mathematical formulae, whether in the text or displayed. For instance, the solidus and the horizontal bar notation for fractions occur, apparently at random, in both cases. There is no sign of war-time stringency in either the quality of the paper or the size of type, but pages and many " leads " could have been saved by attention to detail, and the appearance of the pages would have been improved.

And so on. The author commenced his manuscript in 1931. Considering the author's manifest desire to write a really helpful book, it is a pity that he was so rushed or so uncritical as to leave in his MS. and in proof so many of these blemishes which mar an effort which could easily have been almost entirely admirable.

W. G. B.

Practical Mathematics. (Second edition.) By L. Toft and A. D. D. McKay. Pp. viii, 612. 15s. 1942. (Pitman)

The first edition of this book was reviewed in the *Gazette* for July 1932 (Vol. XVI, p. 220). The present edition is mainly a reprint, although some additional matter has been inserted into the chapter on Linear Differential Equations, and exercises (mostly drawn from recent examination papers of the University of London) have been added both to the collections at the end of the chapters and in a new section at the end of the book. These additions will be welcomed by users of the book, and otherwise there is no reason to alter the opinions expressed in the review of the first edition.

W. G. B.

Air Navigation Plotting. By Mansel Williams. Pp. 88. 2s. 6d. 1942. (Bell)

This is not another book for the beginner who has no previous knowledge of navigation, and as such does not attempt to cover the entire groundwork of the subject in a short space. It does, however, deal with all the D.R. plotting likely to be encountered at an Initial Training Wing or Elementary Flying Training School, or while studying for Part II of the A.T.C. Proficiency Examination. The earlier pages should be useful for Part I of the latter examination. The text is not overburdened with explanatory matter, which even at its best cannot take the place of an instructor, although it must be admitted that sometimes the author is hampered by the brevity which he has imposed upon himself. The most valuable contribution of this book to navigation training is its wealth of practical examples to be worked by the student (30 pp.) and the worked examples (18 pp.) which are set out with extreme clarity and provide excellent models for the pupil. The subject is dealt with from an essentially practical standpoint, and the notes and hints give invaluable assistance to the pupil in understanding and overcoming the

difficulties which are likely to be encountered in the air. The author also has a gift for anticipating the mistakes a pupil is likely to make.

The Air Plot method for finding wind velocity and D.R. position is given detailed treatment, and its importance explained and emphasised. A short section on navigation from the pilot's point of view (Pilot Navigation) is a welcome addition to the usual work covered in a book of this type. Other sections deal with vectors, triangle of velocities, drift, position lines and fixes.

The study of boat problems provides an effective introduction to the triangle of velocities, because the ideas involved make a more direct intuitive appeal to those whose flying experience is limited or non-existent. In the reviewer's opinion, however, it would have been preferable to introduce the triangle of displacements in which the sides of the triangle represent distances gone in one hour or any other convenient time unit. This method also serves as a direct approach to the Air Plot, and the transition to the triangle of velocities is found to be quite simple even for backward pupils.

The few minor faults to be found with this publication do not detract from its general usefulness. On p. 12 an attempt is made to define "angle", although a knowledge of variation and deviation has already been assumed. On p. 13 the term "circular measure" is used in an unfamiliar way to denote an arcual distance in terms of the angle in degrees which it subtends at the centre of the circle: "Angular measure", which is the alternative name suggested, would be preferable. The footnote on p. 25 (which refers to a statement first made on p. 16, where it should appear) contains the somewhat cryptic phrase—"if the triangle be transformed into a circle". On p, 34, definitions of "pin-point" and "fix" are needed, and on p. 54 the misleading term "reciprocal bearing" is used, but not defined: "reverse bearing" is a better phrase which is used in the latest edition of A.P. 1234.

All navigation instructors should find this book useful for the numerous plotting examples contained therein, and every prospective pilot or navigator, whether training with the A.T.C. or R.A.F., could with advantage use it for revision purposes to supplement the instruction received in this subject.

The general arrangement and type are of a high standard, and only one misprint of minor importance was noticed. K. R. I.

Practical Mathematics for the A.T.C. By T. H. WARD HILL. Pp. 120. 2s. 6d. 1941. (Harrap)

This book covers more ground than the A.T.C. syllabus requires, but the additions are not formidable or useless. The arithmetic is adequate, and the exercises have the right aeronautical flavour. The treatment of algebraical formulae on the other hand is sketchy: the exercises are few and are not well graded. There are no exercises on the reading of graphs other than those which the reader is asked to draw for himself. The solution of triangles is covered both by drawing and by simple trigonometry, and the exercises are numerous and practical. Tables are provided, but there are no revision or test papers.

The definitions given of scalar and vector quantities are misleading. "A scalar quantity is one which is only given in magnitude. A vector quantity is one which is given in magnitude and direction." These the author illustrates by saying that a force of 20 lb. is a scalar if its direction is unspecified, but a vector if it is known to be acting (say) vertically downwards! Perhaps this was a loyal attempt to support the Air Ministry, whose original syllabus in mathematics contained, through a copyist's error, the phrase "Vector and Scalar Velocities!"

The production is on the luxuriant side for war-time, with large print and wide margins. The colloquial style and the pen-and-ink sketches will increase its bookstall appeal. A. P. R.

Simple Mathematics. By R. C. SMITH. Pp. 54. 1s. 6d. 1941. (Cambridge University Press)

This little book is for R.A.F. " Selection Board " candidates whose ages are between seventeen and thirty-one. The author has a really simple and readable style, and gives useful hints and warnings. The terms " equation " and " formula " are used indiscriminately as in " changing the subject of an equation ". The exercises are not numerous, but include practice papers.

A. P. R.

Arithmetic for the Services. By D. B. PEACOCK and W. H. DAVEY. Pp. 96. 2s. 1941. (Cambridge University Press)

This is an expansion, for aircraft and artificer apprentices, of a section of an earlier work by the same authors. The subject-matter is mainly mensuration ; chapters on simple interest, profit and loss, rates and bankruptcy seem a little out of place in a work for the Services. The book consists chiefly of exercises, with brief notes and worked examples : there is one page of exercises on graphs, straight line only. There is a good set of test papers of actual Service questions.

A. P. R.

Polish up your Mathematics. By R. C. FAWDRY. Pp. viii, 194. 5s. 1942. (Bell)

The title is misleading as this is nothing so dull as the usual revision course. For instance there are no examples except those worked in the text. Again, Mr. Fawdry sees that you have some mathematics to brush up before applying the polish. There is plenty of solid instruction as well as historical and practical illustration.

The chapter headings give an idea of the ground covered. The Beginnings of Mathematics. Generalised Arithmetic. Angles. Triangles. Circles. Areas and Volumes. Trigonometry. Graphical Representation. Financial Arithmetic. But turn to a few of the section titles for a better idea of how Mr. Fawdry has set about his subject. Why are there sixty seconds in a minute ; how the shop girl gives the correct change ; multiplication by the Romans ; multiplication in the middle ages ; fractions, vulgar and otherwise ; " the damned dots " (so Randolph Churchill referred to decimal points, his only admitted cause of worry as Chancellor of the Exchequer) ; to find the size of the earth ; why the rain beats in your face ; submarine *v.* destroyer ; how Thales measured the height of the Great Pyramid ; design for a window ; gear of a bicycle ; right ascension and declination ; painting the dustbin ; airspace and ventilation ; a problem for gunners.

The text is full of interesting and amusing information :

" This fact has received poetical treatment as we surprisingly discover in Dante's *Paradiso*, Canto II, 97–105. Beatrice explains that if two mirrors are equidistant from a light placed behind the spectator, and a third between them but further away, then the brightness of all three mirrors will be the same."

" At one time the Philippine Islands kept Sunday on the day which was Monday in Borneo. This was due to the fact that the Philippines were discovered by Spaniards who came from the east round Cape Horn whereas Borneo was discovered by Portuguese coming from the west."

" In 1494 Pacioli remarked ' If a man can divide well, everything else is easy '."

Although written primarily for those whose schooldays are over, this lively and instructive book would make a good prize for a boy or girl still at school.

R. C. L.

Practical Mathematics. By C. V. DURELL. Pp. vii, 176, xv. 3s. 3d. 1942. (Bell)

"The object of this volume is to present concisely those processes and principles of elementary mathematics which are an essential part of the equipment of all technical students and form the necessary foundation for more specialised work. It is intended for those who cannot afford the time to study detailed textbooks which deal in turn with arithmetic, algebra, geometry, trigonometry and mechanics."

Once again Mr. Durell and his publishers have produced a good textbook. A very large ground is covered and nothing essential is omitted. The pace is hot, but a normally intelligent and keen class of young men who know no more than elementary school mathematics should find most of the book quite digestible. The more indigestible sections are the Algebra and the Mechanics. Although every teacher must be filled with admiration for Mr. Durell's uncanny knack of presenting so much information so easily and so accurately in these sections where the new ideas and operations are strange and difficult, the treatment is too hasty to be of much permanent value.

Those who value a textbook primarily as a collection of exercises will specially appreciate the care with which these have been selected and graded. Tables and answers are included. R. C. L.

Air Navigation for Beginners. By A. W. SIDDONS. Pp. 32. 1s. 1942. (Arnold)

This is an excellent little book which covers the A.T.C. syllabus up to the standard of the Part I Proficiency Test. The text is a model of accuracy and conciseness, and the diagrams are clear and well-printed. There are many exercises and answers to these may be obtained by instructors from the publishers for the cost of postage.

"Throughout the book an attempt has been made to develop the pupil's intelligence and not to teach him tricks and mnemonics just to get the right answer." Though no one could object to the fully-fledged navigator using rhymes as they surely save thinking, Mr. Siddons is undoubtedly right to suppress them in this introductory book. At this stage it is important that cadets should be able to think things out from first principles. To have got a cadet who is incapable of dealing with, say, deviation and variation without rhymes through an examination which helps to qualify him as a pilot-observer, is something I should not like to have on my conscience. R. C. L.

1417. I checked our height. We were at ten thousand feet. We must have viewed the geometrical limit of vision, the distance at which a straight line makes tangent with the round earth. It struck me as curious how the more distant places—the islands of Borkum, Merkel and Vlissingen, the Isle of Wight and the North Foreland—appeared tilted up at an angle, while the shipping in the Channel looked as though it was sailing in the sky. We felt as though we were flying in the centre of a huge balloon with the sky and sea and earth painted on the outside of its transparent but all-enclosing sphere. . . . Such a scene is the privilege of the airman alone.—From *Vain Glory*, pp. 423, 424, a miscellany of the Great War, edited by Guy Chapman. [Per Mr. P. J. Harris.]

1418. Much no doubt depends on the subject. π is still 22/7, Hitler or no Hitler. . . . Such, no doubt, are the eternal verities whose existence we remarked on above, and whose continuance gladdens the hearts of mathematicians and scientists.—*Times Educational Supplement*, February 10, 1940. [Per Mr. G. L. Parsons.]

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THE UNIVERSITY PRESS, GLASGOW

THE LIBRARY.

THE COPSE, SONNING-ON-THAMES.

MEMBERS preparing to send books to salvage dumps are asked by the Librarian to consider whether some of the mathematical books so unearthed might not merit preservation in the Library. A list of titles, if possible with dates of publication and number of edition, sent to the Librarian, might enable him to indicate which volumes would be welcome.

BOOK WANTED.

D. HILBERT and S. COHN-VOSSEN: *Anschauliche Geometrie.* 1932. (Springer, Berlin.)

Any member who has a copy of this book for sale is asked to communicate with

J. H. Cadwell, Byrdelle, Highdown Hill Road, Emmer Green, Reading.

PLYMOUTH AND DISTRICT BRANCH.

DESPITE " blitzes ", evacuation, and destruction of records, the Branch plans to continue its full activities.

During the spring and early summer of 1942 it is hoped to hold meetings at Exeter and St. Austell as well as in Plymouth.

All members of the Association will, of course, be welcomed, and those who have not received details of the programme but wish to do so are asked to write to the Secretary.

[Revised address:

H.M. Dockyard School, Devonport,

(at) Montpelier, Plymouth.]

F. W. KELLAWAY, Hon. Sec.

SOUTHAMPTON BRANCH.

THE Hon. Secretary would be glad to hear from past members who will be able to attend meetings of this Branch during the summer. A circular addressed to pre-war addresses brought very few replies, most of our members having left Southampton together with their schools.

Any suggestions as to meetings elsewhere than in Southampton would be welcomed.

D. PEDOE.

University College, Southampton.

VICTORIA BRANCH.

REPORT FOR THE YEARS 1940 AND 1941.

THE energies of the Victorian Branch have been entirely diverted to the war effort. In consequence, there are only two meetings to report. 1940, April 15, Annual Meeting; Mr. D. K. Picken (President) and all other officers were re-elected, and Mr. A. McDonell was appointed as an additional committeeman. (Mr. H. M. Campbell later became Librarian, and the Library was removed to the University High School.) At this meeting, addresses commemorating the life and work of the late Honorary President, Professor J. H. Michell, F.R.S., were delivered by the President, Associate Professor Belz, Mr. Barnard and Professor Cherry. It was decided to collate Professor Michell's

lecture notes in a form suitable for preservation and possible publication. 1940, June 24, Mr. A. R. Hutchinson spoke on "Proportional Representation", and in the discussion which followed, references were made to the work on electoral systems performed by the late Professor Nanson.

On June 12 (1940), the Mathematical Association, having offered its services in the pre-enlistment training of Air Crew Reservists in the Empire Air Training Scheme, a committee was set up by the R.A.A.F. Recruiting Drive Committee from representatives of the Mathematical Association, Victorian Teachers' Union, Incorporated Association of Registered Teachers of Victoria, and the Institution of Engineers, Australia, Melbourne Division. Practically all members of this Branch joined with "R.A.A.F. HONORARY INSTRUCTORS". These were of both sexes and included teachers, engineers, actuaries, lawyers, surveyors, municipal officers, opticians and others, "active" or retired, all over the State, 600 strong. Working up to six nights a week, at 160 centres, they trained, in the first year, over 3000 potential air crews. Hundreds of the trainees are now in action overseas. We also co-operated with the Melbourne Swimming Club and the Commonwealth Postmasters' Association in teaching the Reservists swimming and Morse telegraphy. Precise details are not allowed to be published, but no effort has been spared to make this completely voluntary and honorary work a success. In consequence, Branch meetings have been suspended until the Committee decides otherwise.

The address of the Secretary, by courtesy of the Registrar of the University, will henceforth be c/o The Registrar, The University, Melbourne, N. 3, Victoria.

D. K. PICKEN, President.
F. J. D. SYER, Secretary.

BOOKS RECEIVED FOR REVIEW.

H. S. Carslaw and **J. C. Jaeger**. *Operational methods in applied mathematics.* Pp. xvi, 264. 17s. 6d. 1941. (Oxford)

J. D. Chalmers. *Geometrical drawing for beginners.* I. Pp. 32. 1s. 1940. II. Pp. 40. 1s. 3d. 1941. (Craig and Wilson, Glasgow)

S. Chapman. *Edmond Halley as physical geographer and the story of his charts.* Pp. 15, with 6 plates. 2s. 6d. 1941. (Royal Astronomical Society)

A. Cohen. *Elements of calculus.* Pp. v, 583. 10s. 6d. 1941. (Heath, Boston; Harrap)

G. S. Diwan and **D. S. Agashe**. *Differential equations.* Pp. x, 316. N.p. 1941. (St. Xavier's College, Bombay)

S. L. Green. *Algebraic solid geometry. An introduction.* Pp. vii, 133. 6s. 1941. (Cambridge)

A. Hooper. *A mathematical refresher.* Pp. viii, 308. 9s. 6d. 1941. (Oliver and Boyd, Edinburgh)

J. G. Lockhart and **T. M. Young**. *Sure Foundation arithmetic.* Book 3: pp. 104. 1s. 3d. 1941. Answers; pp. 48. 1s. Book 4: pp. 159. 1s. 4d. 1940. Answers: pp. 64. 1s. Book 5: pp. 176. 1s. 6d. 1940. Answers: pp. 64. 1s. Book 5A: pp. 96. 2nd impression. 1s. 1941. Answers: pp. 25. 9d. (Craig and Wilson, Glasgow)

N. McArthur and **A. Keith**. *Intermediate algebra.* Pp. x, 356. 8s. 6d. 1942. (Methuen)

D. B. Peacock and **W. H. Davey**. *Arithmetic for the Services.* Pp. 96. 2s. 1941. (Cambridge)

C. Singer. *A short history of science to the nineteenth century.* Pp. xiv, 399. 8s. 6d. 1941. (Oxford)

R. C. Smith. *Simple mathematics. A revision course for R.A.F. Selection Board candidates and others.* Pp. 54. 1s. 6d. 1941. (Cambridge)

EDITORIAL NOTE.

THE need for strict economy in the use of paper means fewer pages per issue of the *Gazette*. In order that the actual content may avoid undue diminution, Articles are now being printed in a smaller fount of type and Notes also will in future be set up in this size. The slight decrease in margin is an additional off-set to the fewer pages. As, however, a great many Notes were already set up in the larger fount of type when the recent paper restrictions came into force, these Notes must be used up before a change in the fount used for Notes can be made ; this accounts for the mixed appearance of the present issue. It also accounts for what will be a considerable delay in the publication of a large number of Notes now in hand. We can only offer apologies to the many valued contributors whose work is thus held back.

THE LIBRARY.

THE COPSE, SONNING-ON-THAMES.

WILL Mr. **A. G. Champernowne** and Mr. **D. Sholl** please communicate with the Librarian? Any reader in touch with either of these members will do the Association a service by calling attention to this appeal.

LONDON BRANCH.

REPORT OF MEETINGS, 1942.

THREE meetings have been held so far this year.

14th March. Brains Trust. A series of questions previously collected from members was submitted to the Trust, which consisted of all the members present. A valuable interchange of opinion resulted.

2nd May. Mr. K. R. Imeson gave a talk on " A.T.C. Training and its relation to school mathematics ". He outlined the aims of the course and illustrated it by details of methods used in navigation and the interpretation of some formulae of the theory of flight.

30th May. Mr. S. Inman explained the construction of a table of logarithms. Elementary processes only were used, by means of which the logarithms of 24 two-figure primes were calculated. From these the table was completed. Interpolation was used to obtain the logarithms of four-figure numbers. A second method depending on repeated square roots was also given and the history of the subject was sketched.

The Hon. Secretary of the Branch would be most grateful for offers of papers or suggestions of subjects for meetings to be held next session ; such offers and suggestions should be sent to Dr. A. J. Taylor, 30 Manor Gardens, Purley, Surrey.

A. J. TAYLOR, Hon. Sec.

NEW SOUTH WALES BRANCH.

REPORT FOR 1941.

THE membership of the Branch now stands at : full members, 21 ; associate members, 144 ; total, 165.

Three Ordinary meetings were held during the year and one meeting of the Organising Committee. At the Ordinary meetings the following addresses were given :

(i) Addresses and discussions on three teaching topics ;

(*a*) the teaching of graphs, introduced by Mr. Austin ;

(*b*) the arrangement of work in the solution of a triangle in trigonometry, introduced by Mr. Hallett ;

(*c*) a sequence of topics in trigonometry for the non-specialist, introduced by Brother Liguori.

This proved a very interesting and instructive evening.

(ii) An address by Mr. N. A. Esserman on the work of the National Standards Laboratory. This was an informative address, illustrated by diagrams, models and some of the apparatus used in the Laboratory.

(iii) The Annual meeting, at which reports from the Hon. Secretary, the Hon. Treasurer, and the Director of the Problem Bureau were read. The report on the work of the Problem Bureau showed that 25 problems had been sent in ; solutions to 24 had been returned to the propounders.

The office-bearers for 1942 were elected as follows : *President*, Mr. D. J. Austin ; *Vice-Presidents*, Professor Wellish, Mr. A. A. Thorne, Dr. I. S. Turner ; *Joint Hon. Secretaries*, Miss E. A. West, Mr. H. J. Meldrum ; *Hon. Treasurer*, R. J. Gillings ; *Director of the Problem Bureau*, Mr. R. J. Gillings.

H. J. MELDRUM, Hon. Secretary.

BOOKS RECEIVED FOR REVIEW.

A. Bakst. *Mathematics. Its magic and mystery.* Pp. xiv, 790. 21s. 1941. (Chapman and Hall)

W. Briggs and **G. H. Bryan.** *Tutorial algebra. II.* 5th edition. Revised by G. Walker. Pp. viii, 559. 12s. 6d. 1942. (University Tutorial Press)

W. B. Coulthard. *Transients in electric circuits, using the Heaviside operational calculus.* Pp. viii, 203. 25s. 1941. (Pitman)

F. E. Croxton and **D. J. Cowden.** *Applied general statistics.* Pp. xviii, 944, xiii. 20s. 1941. (Pitman)

C. V. Durell. *Practical mathematics.* Pp. viii, 176, xv. 3s. 3d. 1942. (Bell)

R. C. Fawdry. *Polish up your mathematics.* Pp. viii, 194. 5s. 1942. (Bell)

G. A. Ferguson. *The reliability of mental tests.* Pp. x, 150. 6s. 1942. (University of London Press)

K. Käfer. *Der Kettensatz. Ein Beitrag zur Geschichte und Theorie des kaufmännischen Rechnens.* Pp. 415. 8 francs (Swiss). 1941. Mitteilungen aus dem handelswissenschaftlichen Seminar der Universität Zürich. Neue Folge, Heft 68. (Schulthess, Zürich)

B. B. Low. *Engineering mechanics.* Pp. vii, 252. 12s. 6d. 1942. (Longmans)

L. Malone. *Post-certificate mathematical tests.* Pp. 85. 2s. 1942. (Macmillan)

G. Peacock. *A treatise on algebra. I. Arithmetical algebra.* Pp. xvi, 399. *II. On symbolical algebra and its applications to the geometry of position.* Pp. x, 455. 1940 ; reprinted from the edition of 1842-5. $6.50. (*Scripta Mathematica*, Yeshiva College, New York)

A. W. Siddons. *Air navigation for beginners.* Pp. 32. 1s. 1942. (Arnold)

T. M. Spaulding and **L. C. Karpinski.** *Early military books in the University of Michigan Libraries.* Pp. xvi, 45 ; 37 plates. $2. 1941. (University of Michigan Press, Ann Arbor, Michigan)

Sir D'Arcy Wentworth Thompson. *On growth and form.* New edition. Pp. 1116. 50s. 1942. (Cambridge)

L. Toft and **A. D. D. McKay.** *Practical mathematics.* 2nd edition. Pp. vii, 612. 15s. 1942. (Pitman)

H. H. Wolfenden. *The fundamental principles of mathematical statistics.* Pp. xiii, 379. $5. 1942. (Macmillan Company, Canada)

Portraits of famous physicists. With biographical accounts by Henry Crew. 12 plates. $3.75. 1942. (*Scripta Mathematica*, Yeshiva College, New York)

Table of squares of all numbers less than 300. 1s. 1942. (Scientific Computing Service, 23 Bedford Square, London, W.C. 1)

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CONTENTS.

	PAGE
ISAAC NEWTON, 1642–1727,	201
BLITZED LIBRARIES: AN APPEAL,	201
ON SOME GEOMETRICAL INEQUALITIES. D. PEDOE,	202
COMMITTEE ON CURRICULUM AND EXAMINATIONS,	208
SOME HOMOTHETIC TRIANGLES RELATED TO THE EULER LINE. R. H. COBB,	209
MEMORANDUM ON POST-WAR EDUCATION AND TRAINING OF PHYSICISTS,	212
MATHEMATICAL NOTES (1627–1635). T. W. CHAUNDY; H. F. EVANS; R. L. GOODSTEIN; S. INMAN; J. PEACOCK; A. W. SIDDONS and H. WEBB; H. P. F. SWINNERTON-DYER,	214
REVIEWS. W. G. BICKLEY; C. D. BROAD; G. A. CLARKSON; N. M. GIBBINS; K. R. IMESON; E. H. LOCKWOOD; R. C. LYNESS; J. C. P. MILLER; A. P. ROLLETT; F. SANDON; C. O. TUCKEY,	223
GLEANINGS FAR AND NEAR (1413–1418),	208

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(7) REPORT ON THE TEACHING OF ELEMENTARY MATHEMATICS IN GIRLS' SCHOOLS, 1916, 1s. net. [Replaced by (20).]

(8) REPORT ON THE TEACHING OF MECHANICS (*The Mathematical Gazette*, No. 137, Dec. 1918), 1s. 6d. net. [Now replaced by (22).]

(9) REPORT ON THE TEACHING OF MATHEMATICS IN PUBLIC AND SECONDARY SCHOOLS, 1928, 1s. 6d. net. [Out of print.]

(11) REPORT ON THE TEACHING OF MECHANICS IN GIRLS' SCHOOLS, 1923, 6d. net. [Out of print.]

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(14) REPORT ON THE TEACHING OF MATHEMATICS TO EVENING TECHNICAL STUDENTS, 1926, 1s. net.

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(28) A SECOND REPORT ON THE TEACHING OF GEOMETRY IN SCHOOLS, 1938; 2s. 6d. net.

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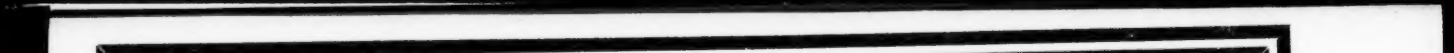
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